ON PERFORMANCE BASED DESIGN OF SMOOTH SLIDING MODE CONTROL

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DECLARATION

It is declared that this is an original piece of my own work, except where otherwise acknowledged in text and references. This work has not been submitted in any form for another degree or diploma at any other university or institution for tertiary education and shall not be submitted by me in future for obtaining any degree from this or any other University or Institution

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ABSTRACT

The Sliding Mode Control (SMC), being famous for remarkable robustness, uses a discontinuous controller and is established in two phase, namely, reaching phase and sliding phase. The sliding phase, represented by reduced order dynamics, offers certain benefits like invariance with respect to parameters and disturbances. However, the discontinuous nature of the controller and imperfection of physical systems imposes the undesirable high frequency oscillations called chattering. In addition, the reaching phase has been reported to be sensitive to uncertainties and disturbances which may degrade the performance or even cause stability problems in some sensitive applications.

The Smooth SMC (SSMC), known for chattering eradication, do not approximate the actual sense of sliding modes. In addition, the Integral SMC (ISMC) eliminated the reaching phase and hence any sensitivity to any unwanted phenomenon in the reaching phase. However, the SSMC such as Smooth Super Twisting Algorithm (SSTA) has no theoretical measures for the performance and/or robustness while the ISMC still suffers due to chattering, though reduced, and loses parameter invariance property due to no order reduction in the sliding phase.

In this thesis, a novel Lyapunov function based analysis of the SSTA is proposed and by the virtue of stability analysis, novel performance and robustness parameters are developed which include, analytical expressions for choosing the gains of the controller, settling time of the closed loop system and stability bounds for a class of uncertainties. The proposed settling time formulation suggests a methodical approach to SSTA design in contrast to the available rules of thumb. The proposed design framework is validated against a challenging problem of the Underground Coal Gasification (UCG) process control.

On the other hand, the ISMC has been investigated for chattering removal and possible performance degradation due to parametric variations in the sliding phase. In this regard, the discontinuous part of the ISMC is made **smooth** to eliminate chattering and the continuous part of the controller is proposed as a Linear Matrix Inequality (LMI) based LPV gain scheduling state feedback controller which deals with the possible performance degradation due to parametric variations. The results are proved mathematically and are validated experimentally on laboratory test bench ball on a beam balancer.

Keywords: Sliding Mode Control (SMC), Smooth Super twisting Algorithm (SSTA), Higher Order Sliding Mode (HOSM), Discontinuous System, Underground Coal Gasification (UCG), Lyapunov Function, Non-Vanishing Perturbations, Integral Sliding Mode Control (ISMC), Linear Parameter Varying (LPV), Linear Matrix Inequality (LMI), Ball on a Beam Balancer (BBB).

LIST OF PUBLICATIONS

Journal Publications

1. ***Imran Khan***, *A. I. Bhatti, A. A. Uppal and Q. Khan.* "Robustness and Performance Parameterization of Smooth Second Order Sliding Mode Control". International Journal of Control, Automation and Systems), DOI : 10.1007/s12555-014-0181-6, Vol. 14, no. 3, 2016.

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Conference Publications

1. ***Imran Khan***, *A. I. Bhatti, Q. Khan and Q. Ahmad*, "Sliding mode control of lateral dynamics of an AUV", 9th International Bhurban Conference on Applied Sciences and Technology (IBCAST), 2012.

2. ***Imran Khan***, *A. I. Bhatti, Qudrat Khan*, "An Integral Sliding Mode Control Methodology for Systems with Linearly Varying Parameters in the Sliding Mode", 5^{th} World Engineering Congress, NUST Islamabad, 2013.

3. ***Imran Khan***, *A. I. Bhatti, Q. Khan and R. Anjum*, "Constructive Mechanism of Lead/Lag Design for a Class of Linear Systems", International Conference on Emerging Technologies (ICET), 2016.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

LIST OF ACRONYMS

SMC	Sliding Mode Control
LTI	Linear and Time Invariant
SISO	Single-Input-Single-Output
ODE	Ordinary Differential Equations
MIMO	Multiple-Input-Multiple-Outputs
PID	Proportional + Integral + Derivative
LQR	Linear Quadratic Regulator
HOSM	Higher Order Sliding Mode
STA	Super Twisting Algorithm
STO	Super Twisting Observer
SSTA	Smooth Super Twisting Algorithm
RTA	Real Twisting Algorithm
SRTA	Smooth Real Twisting Algorithm
SSOSM	Smooth Second Order Sliding Mode
SOSM	Second Oder Sliding Mode
ISMC	Integral Sliding Mode Control
SISMC	Smooth Integral Sliding Mode Control
SSMC	Smooth Sliding Mode Control
FOSM	First Order Sliding Mode
VSS	Variable Structure System
VSC	Variable Structure Control
SMO	Sliding Mode Observer
SOSMO	Second Order Sliding Mode Observer
UCG	Under Ground Coal Gasification
LPV	Linear Parameter Varying
LMI	Linear Matrix Inequality
BBB	Ball on a Beam Balancer
PDE	Partial Differential Equation
ALE	Arithmetic Lyapunov Equation
CCT	Combined Cycle Turbine
C	Carbon
O_2	Oxigen
H_2	Hydrogen
N_2	Nitrogen
CO	Carbon Mono oxide
CO_2	Carbon dioxide
H_2O	Steam
CH_4	Methane

LIST OF SYMBOLS

Symbol	**Description**	**Units**
t_s	Settling Time	Second
t_r	Rise Time	Second
M_p	Overshoot	-
σ, S, s	Sliding Surface	-
r	Reference Input	-
x	State Vector	-
x_i	i^{th} State	-
x_0	Initial Conditions	-
t	Time	Seconds
u_{eq}	Equivalent Control Component	-
$V(t, x)$	Lyapunov Function	-
k_i	i^{th} gain of a controller	-
ρ	Smoothing Parameter	-
$\zeta(t, x)$	Perturbations	-
A	System Matrix	-
$\lambda_{max}[\cdot]$	Maximum eigen value	-
$\lambda_{min}[\cdot]$	Minimum eigen value	-
T_s	Settling Time	Seconds
$.^T$	Transpose	-
ℓ	Invariant Set	-
L	Perturbation upper bound	-
T_{ps}	Settling Time of perturbed system	Seconds
χ_c and ψ_c	Center coordinates of an ellipse	-
Ω	Manifold	-
M_{coal}	Molecular weight of coal	$g/mole$
M_{char}	Molecular weight of char	$g/mole$
$a_{i,j}$	Stoichiometric coefficient of i^{th} specie in j^{th} chemical reaction	-
r_j	Reaction rate of j^{th} reaction	-
C_s	Total solid phase heat capacity	$Cal/K.cm^3$
ht	Convective heat transfer coefficient	$Cal/sec.K.cm^3$
T	Ignition Temperature	K
H_s	Solid phase heat source	$Cal/sec.cm^3$
mf_i	Mole fraction of i^{th} gas	-
H_i	Heat of combustion of i^{th} gas	KJ/mol
u	Reactor input (flow rate of inlet gases)	$mol/cm^2.sec$
h	Calorific value of the product gases	KJ/mol
$A(\rho)$	Open loop parameter dependent system matrix	-
$A_{cl}(\rho)$	Closed loop parameter dependent system matrix	-
v	Eigenvector	-
$.^*$	Complex conjugate	-

$M(\rho)$	Scheduled gain	-
L	Beam length	cm
r	Liver arm offset	cm
θ_l	Servo gear angle	deg

Chapter 1

INTRODUCTION

"And whoever fears ALLAH - He will make for him a way out and will provide for him from where he does not expect. And whoever relies upon ALLAH - then He is sufficient for him. Indeed, ALLAH will accomplish His purpose. ALLAH has already set for everything a [decreed] extent."

Al–Quran, 65: 2-3.

The tradition of research in control theory is very long and distinguished which stretches back to nineteenth century dynamics and stability analysis. It aroused as an engineering discipline in late 1950s. Nyquist, Bode, Evan and Wiener were the pioneers who worked in frequency domain for dynamical analysis and controller synthesis. The early control system design techniques fulfilled the cause of automation but robustness, computational complexity when dynamics gets complex and many other accompanied limitations were still open research challenges for the control system community.

A number of control system algorithms were developed to provide robust automation of dynamical systems. These include H_∞ and μ-synthesis based loop shaping, Linear Matrix Inequality (LMI) based controllers, adaptive controllers, output feedback linearization, back stepping and Sliding Mode Control (SMC). Among these, the SMC got popularity due to its simplicity, remarkable properties of robustness against uncertainty/disturbances of matched nature[1] and applicability to both linear and nonlinear systems.

[1] SMC that cope with disturbances of mismatch nature are also researched but are not discussed here.

1.1 Sliding Mode Control Theory and Dynamic Systems

In the mid-twentieth century S. V. Emalyanov and V. I. Utkin, the Russian theoreticians, and their co-workers felt that the conventional state feedback techniques were not robust against disturbances and nonlinearities. Emalyanov and company came up with the idea of having a special type of state feedback controller with an additional property of polarity (direction or gain) switching. They called it the **"Variable Structure Control (VSC)"**, now-a-days known by the name Sliding Mode Control (SMC) [2, 3, 4, 5].

The basic idea of SMC is to enforce ***sliding modes***, in a pre-defined manifold known as the sliding manifold, sliding surface, switching line or hyperplane, in a given system's state space, with the application of a discontinuous (switching) controller.

Traditionally, SMC occurs in two phases. The time instant, when the system's states trajectories are forced, from an initial condition to a pre-defined sliding manifold, is known as the reaching phase. The accomplishment of reaching phase is followed by a special type of system motion/trajectories, known as the sliding phase. In this phase the system's states trajectories are restricted to stay on the sliding manifold and are allowed to ***slide*** along the surface to an equilibrium (trivially the origin). The phenomenon of sliding phase and hence the existence of sliding modes is a bench mark property of SMC as it provides guaranteed robustness against model imperfections, parametric variations and certain class of uncertainties and external disturbances (usually matched and bounded) [6, 7]. However, the discontinuous controller has to switch with a very high frequency (theoretically infinite frequency), about the sliding manifold, in order to force the states trajectories confined to the sliding manifold. This phenomenon, which is too treacherous for mechanical actuators and the underlying mechanical system to be controlled, is known as chattering. In addition to chattering an SMC is sensitive to disturbances and uncertainties in the reaching phase.

A number of methods were proposed to overcome these problems e.g., the Higher Order Sliding Mode (HOSM) control tackled the issue of chattering while the Integral Sliding Mode Control (ISMC) eliminated the reaching phase and hence they gave a strong argument of handling and assuming **a dynamical system to be nominal**. Each of these variants are subjected to some trade-offs, which will be discussed in comprehensive details in the forthcoming chapters.

1.2 Motivation of the Work

The tendency of using SMC algorithms, for the control and observation purposes, steered the mathematicians and engineers to investigate these algorithms for all possible improvements. As mentioned earlier, the HOSM was invented with the rationale to keep intact the quality and properties of the conventional SMC and yet to reduce chattering. The HOSM algorithms performed well regarding chattering reduction but they were still sensitive to un-modeled fast dynamics. The rationale of complete chattering free control brought the concept of Smooth Sliding Mode Control (SSMC) algorithms. The SSMC algorithms provided a continuous control action but at the cost of robustness. The issue of robustness is then solved using sliding modes based observers. The SSMC algorithms, in combination with the disturbance observers, proved very effective in terms of robustness and chattering reduction. However, the SMC algorithms whether, first order or higher order and/or smooth, are mathematically synthesized to ensure robustness. The algorithm gains are often claimed greater than the upper bound of the possible matched disturbance for this purpose. Thus, a concrete mathematical formulation is necessary, which answers all the questions regarding the effects of various controller parameters on the closed loop performance of the underlying system.

Moreover, if no observer is used for the robustification of the SSMC then what must be the value of controller gains and other parameters to ensure robustness as well as performance. The current research work is focused on the performance based design of the SSMC.

Furthermore, some dynamical uncertain systems are very sensitive to even very small disturbances and uncertainties in reaching phase[2], which may cause undesirable results and in worst cases it may cause instability of the closed loop system. The proposal of the ISMC algorithm [8] sorted out this issue by reaching phase elimination. The reaching phase elimination facilitated the controller construction using a valid assumption of nominal system[3]. Inherently, there is no order reduction and eventually all the system states and parameters appear in sliding mode. This makes the closed loop performance, with ISMC in the loop, sensitive to any possible parametric variations present in the underlying system. This research work strives to find a way such that any possible variations in the system parameters do not degrade the overall closed loop performance. Moreover, the chattering, although it is less than the conventional SMC, is to be smoothen out.

This research however consider the following prime assumptions.

- The uncertainties and disturbances may be vanishing or non-vanishing but they are norm bounded and matched.
- The system output and states are measurable.
- The constant system parameters are precise and the varying parameters are available/measurable.
- The relative degree is one with respect to the sliding manifold.

[2] Sliding phase of any SMC algorithm has guaranteed invariance to disturbances.
[3] The ISMC rejects all the uncertainties/disturbances from the very beginning and hence the system becomes a nominal one.

1.3 Statement of Contribution

This research work presents the following main contributions.

1. The robustness and performance parameterization of the Smooth Super Twisting Algorithm (SSTA) is performed. The robust stability analysis of SSTA contribute the following:
   - The robust stability of the SSTA is carried out using Lyapunov's approach.
   - A methodical way for choosing the SSTA gains, in terms of analytical expressions, is proposed.
   - A systematic and operation oriented mathematical formulation for the convergence time of the closed loop dynamics with SSTA in the loop, is proposed, which may be used for performance enhancement.
   - The consequences of the controller parameters k_1, k_2 and ρ, on the performance and stability of the closed loop system is explored.
   - The SSTA, with the proposed systematic design procedure, is validated on a highly nonlinear and challenging process control problem of the Underground Coal Gasification (UCG).
2. The ISMC is made hybrid with the LMI based Linear Parameter Varying (LPV) controller (as the continuous part of the controller) for a class of uncertain systems which can be expressed in LPV form. This completely eliminate the possible performance degradation due to any parametric variations in the closed loop. In addition, the discontinuous part of the ISMC controller is made smooth to completely eliminate chattering for practical purposes. The hybrid of ISMC and LPV is practically implemented for stabilization purposes of a Ball on a Beam Balancer (BBB), which is a nonlinear uncertain system.

1.4 Overview

A brief overview of the contents contained chapter wise in this thesis is given below.

Chapter 2 explores the historical evolution of the tools and techniques for/in SMC. This chapter focuses the literature which took a VSC to the modern day robust and smooth SMC algorithms. The main theme on which this chapter stands is to give a step by step the evolution of SMC with its merits and demerits. This discussion then led us to the research gap, explored in this thesis.

Chapter 3 highlights the mathematical fundamentals and design procedures of SMC algorithms, which will support the contribution in this thesis. A comparison of linear and nonlinear control techniques is given which forms a road map between the linear and nonlinear control techniques. The design of various SMC controllers is shown via numerical examples and is elaborated using MATLAB/SIMULINK simulations. The chapter ends with problem statement and research objectives.

Chapter 4 presents the first contribution of this thesis. The robust stability and performance of SSTA is parameterized in this chapter. The effect of various controller parameters on the performance and robustness of the SSTA is explained mathematically. In addition, the analytical expressions for the gains of the controller are developed and the algorithm is tested on a challenging process control problem of UCG.

Chapter 5 is comprised of the second contribution of this thesis. The mathematical framework, based on the LMI and LPV control, is developed for the conventional first order ISMC. The proposed framework is proved mathematically and a step by step design procedure is explored. The algorithm is tested on an experimental laboratory test bench BBB to explore practically the algorithm's authority.

Chapter 6 concludes this thesis with emphasize on major contributions. In addition, some future directions are also given which may be considered as a direct consequence of the contributions in this thesis.

Chapter 2

DEVELOPMENT OF SLIDING MODE CONTROL

"A pessimist sees the difficulty in every opportunity; an optimist sees the opportunity in every difficulty."

Winston Churchill.

This chapter presents a historical background of various techniques and algorithms exercised by the Sliding Mode Control (SMC) research community. The evolution of SMC theory from Variable Structure Control (VSC) initiated a long lasting discussion on the SMC theory because it provided properties like parameter invariance and performance in the presence of uncertainties as well as external disturbances. Over a period of time the research community identified various problems and proposed a number of variants of the emerged sliding mode control in order to cope with each identified problem. These variants include Integral Sliding Mode Control (ISMC), Dynamic SMC (DSMC), Dynamic-Integral SMC (DISMC), Terminal SMC (TSMC), Higher Order SMC (HOSMC) and Smooth Sliding Mode Control (SSMC) etc.

A common consideration in the development of each of these variants was to keep intact the basic properties of the conventional SMC theory. The advent of ISMC opened a research direction with the rationale of robustifying some controllers, by reaching phase elimination, while the parameterization of various SMC algorithms was a fascinating advancement in the last decade which allowed further structural improvements of the SMC algorithms. Moreover, the SSMC algorithms were of remarkable importance regarding the practical aspects of implementing

SMC algorithms. However in fact each of these have some limitations which will be discussed partially in this chapter[1]. The chapter is organized as follows:

In Section 2.1, the evolution of the theory of SMC with its merits and demerits is discussed. Section 2.2 explores the utility of ISMC with its advantages and disadvantages. In Section 2.3 various remedies, for chattering reduction, are explored with some details. Section 2.4 introduces a step by step advancement in HOSM algorithms with their fascinations and shortcomings. In Section 2.5 some of the smooth SMC algorithms are presented. Section 2.6 highlights some of the mathematical tools being utilized for the stability analysis and proving finite time convergence of various SMC algorithms. Section 2.7 gives the research directions based on intuition attained in the above sections and Section 2.8 summarizes this chapter.

2.1 Sliding Mode Control

The term variable structure control was introduced by S. V. Emelyanov in 1967. Later on, the existence of the phenomenon known as **sliding mode** was introduced by V. I. Utkin in 1977[2]. The publication of survey paper [3] opened further arenas for the researchers in the control system community and the SMC emerged as a field. The SMC was formally introduced and implemented in [6, 9, 10, 11].

Sliding modes, as a phenomenon, are established in two phases, known as the reaching phase and the sliding phase. In reaching phase, the system state trajectories are forced towards a pre-defined manifold, known as the sliding manifold (σ), by a discontinuous switching controller while in reaching phase the state trajectories **slide** along the surface to an equilibrium.

[1] The water bed effect of control engineering is pronounced in all these variants of SMC. A simplest example of water bed effect in SMC theory is that the performance will be sacrificed if robustness is enhanced and vice versa.

[2] Sliding modes is a special type of system motion which is enforced between two symptomatically different or almost opposite system structures.

SMC offered impressive closed loop performance, which contain invariance with respect to parametric variations, remarkable robustness with respect to disturbances/perturbations of matched type, uncertainties/model-imperfection and importantly, the dynamics during sliding are of reduced order. However, the discontinuous controller has to switch at a very high frequency (theoretically infinite frequency) for *sliding modes* enforcement. However, unfortunately the infinite/very-high frequency can not principally be attained with physical systems. This limitation, in addition to the delays in physical systems, caused the high frequency oscillations, called *chattering*, with respect to the switching line. These high frequency oscillations were hazardous for the actuators and a worst case scenario may be the total control system failure. The relative degree requirement add another restriction on the use of the conventional first order SMC[3]. In addition, the *sliding modes* were invariant to the unwanted phenomena like uncertainties and disturbances but the reaching phase of the conventional First Order SMC (FOSMC) was never claimed to be invariant to these unwanted/unavoidable factors.

The highlighted problems in the FOSMC were of particular interest to the research community and over many years the research aimed at nullifying these problems through certain structural modifications in the FOSMC.

2.2 Integral Sliding Mode Control

In order to guarantee insensitivity to uncertainties/model-imperfections and disturbances the only thing the designers needed was to ensure the **insensitivity**, starting from the initial time instant. In other words, a justification to the **assumption of nominal system** was required. The intended important advantage of this concept was the certification of exact tracking of the trajectories, designed actually for the nominal system, in the non-nominal (original) state space.

[3]The first order SMC is applicable to systems having relative degree **"one"** with respect to the switching manifold i.e., Control appears explicitly in the first total time derivative of the switching manifold.

The only possibility in SMC regimes which attained the above mentioned objective, by eliminating the reaching phase, was the concept of ISMC [12, 8]. The reaching phase elimination was achieved using a special type of sliding surface, known as the integral sliding surface or integral manifold. The dynamics of a system on this surface, which start from the initial instant of time due to ISMC algorithm, is equipped with the following highlighted merits:

- It makes the assumption of nominal system valid, as the uncertainties/disturbances are coped with from the very beginning.
- It needed the smaller gain of the discontinuous controller which leads to chattering reduction[4].

However there are also some drawbacks of ISMC:

- The important property of order reduction is sacrificed, as the sliding modes, established by the ISMC algorithm, have the dimension similar to the original state space. In other words, we can say that the extended state space is comprised of all the dynamics (states) of the system to be controlled.
- The fact that sliding modes comprised the original state space makes the performance of the closed loop system sensitive to parametric variations.
- The ISMC needed to have measurement/estimation of all the system states from the very beginning.
- Chattering, although reduced, but is still an issue in various control applications e.g. a multi-loop control scheme in which an inner loop's continuous controller is driven by a continuous signal from a controller in the outer loop.

[4]Because the discontinuous controller is accompanied by a nominal controller which is supposed to have compensated the nominal system's dynamics.

2.3 Chattering

Chattering was the most pronounced unwanted phenomenon in the earlier innovations of SMC. The experts proposed a variety of tools/variants for/of the conventional SMC, some of which are listed here.

2.3.1 Boundary Layer Approach

The boundary layer approach address the chattering hazard by increasing the physical dimensions of the surface such that the sliding modes are said to be established in a predefined vicinity of the surface rather than exactly at the surface[5]. This scenario is depicted in Figure 2.1 and is achieved by replacing *sign*-function by a *sat*-function [13].

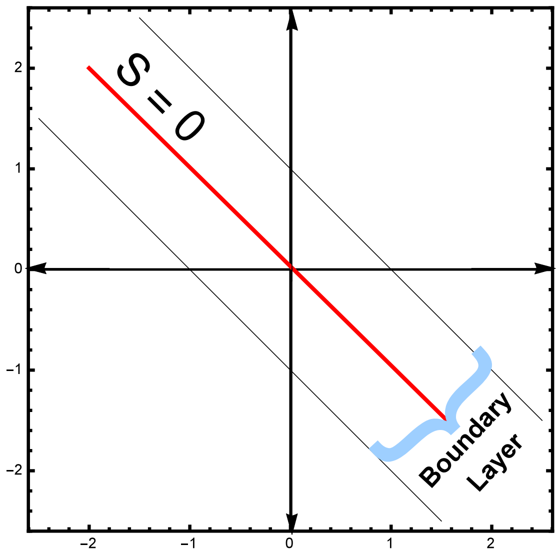


FIGURE 2.1: Boundary Layer approach for chattering minimization

The smoothening of the controller action achieved using this approach is directly dependent upon the thickness/width of the defined vicinity. However, the unavoidable water bed effect comes in action in such a way that increasing thickness/width

[5]This means that switching will not occur as frequent as in the case of strictly enforcing $S = 0$, where S represent the sliding surface.

of the boundary layer causes serious performance degradations. This is exactly because the boundary consideration approximates the actual sense of SMC and hence causes losing the essence of it.

2.3.2 Observer Based Chattering Reduction

The un-modeled/unknown dynamics of a physical process, under consideration, is one of the highlighted cause of producing chattering [7]. This method implements the SMC with an asymptotically convergent observer in the closed loop. The observer is meant to bypass the un-modeled dynamics and hence producing almost ideal sliding mode.

The SMC with these and a number of other tools were appreciated in the research regimes. These techniques solved the problem of chattering to some extent but a drawback was that, the *sliding modes* exhibited under the lights of these techniques were termed as approximate *sliding modes* [14]. In the mean time the researchers carried out the cause oriented efforts to find variants of the conventional FOSM such that the properties of the SMC are kept intact. One such variant equipped with an in-built filter for chattering suppression is known as the DSMC [15]. Another variant with the idea to hide discontinuity of control in its higher derivatives has been realized using HOSMC [16, 17, 18, 19].

2.4 Higher Order Sliding Mode Control

Higher order sliding mode control, being invented in the last decades of the 20^{th} century by the efforts of S. V. Emalyanov and his research colleagues, preserved the main structure of the standard SMC with motion on a discontinuous set of dynamic system understood in Fillipov's sense [20]. Let σ represent the sliding surface. Then, the sliding set is determined by:

$$\sigma = \dot{\sigma} = \ddot{\sigma} = ... = \sigma^{r-1} = 0 \qquad (2.1)$$

where r is the relative degree of the system with respect to the sliding manifold (σ). This sliding set is termed as r^{th} order sliding set and the sliding motion being maintained against this set is called $r-$sliding mode. Furthermore, the sliding order indicates the smoothness degree of the dynamics of the system in a small neighborhood of the sliding mode.

The HOSM control is a generalization of the conventional FOSM control. In conventional FOSM control the switching takes place against the line, plane or hyperplane[6] defined by $\sigma = 0$ while in HOSM control, the switching takes place against the intersection of the lines, planes or hyperplanes defined by $\sigma = 0,\ \dot{\sigma} = 0,\ \ddot{\sigma} = 0,\ ....\ \sigma^{r-1} = 0$. The HOSM algorithms greater than second order, suffered from practical feasibility as the higher is the derivative to determine the greater is the sensitivity to noises[7].

The practical HOSM algorithms reported are of second order such as Super Twisting Algorithm (STA) and Real Twisting Algorithm (RTA). These algorithms gained admiration among the experts in the control community, because of the generalized structures, simpler implementation, enhanced performance and more importantly, reduced chattering[8]. Moreover, the STA is applicable to systems having relative degree one while the RTA can be used for relative degree two systems.

The HOSM algorithms (STA and RTA) minimized the chattering up to some extent by reducing the physical dimensions of the switching manifold but it never claimed *complete elimination* of chattering. This flaw was produced by the fact that the HOSM algorithms are very sensitive to unmodeled fast dynamics [21, 22] and chattering may appear sooner or later in the system. One such situation, where even a small amount of chattering can cause trouble was a multi-loop system,

[6] The switching will take place against a line or plane or hyperplane depending upon the system order and especially the order of the dynamics during sliding.

[7] Practically noises occur in both sensing and actuation. The derivatives of noise signals result in very high peaks. This phenomenon is termed as noise amplification. This amplification is directly proportional to the order of the derivative.

[8] They do not approximate the true sense of SMC.

where STA and RTA are not auspicious with respect to performance improvement, especially when a loop requiring a continuous signal is driven by another loop. Thus the researchers thought of an SMC which was desired to be smooth and differentiable at sampling times. All such controller were known by the name Smooth sliding mode controllers. The smooth control algorithms, due to their continuous (smooth or almost chattering free) control action, were found to be very effective in many sensitive applications.

2.5 Smooth Sliding Mode Control

The *smoothness* of the control law was employed for a *chattering-free* SMC. As stated earlier smoothening of the control signal was obtained by approximate sliding modes (approximation of $sign-$function by $sat-$function which is usually known as the boundary layer approach) or the HOSM. However, neither *approximation* nor *HOSM* were able to satisfy the researchers, regarding ***smoothness***.

In the conventional FOSM the discontinuous controller was given by the following equation:

$$u = -k sign(\sigma)$$

where u is the control input, σ is the sliding surface and k is a constant controller gain selected intelligently to cope with the disturbances and uncertainties. The constant gain causes a sort of inertia when the sliding modes are established and the switching between two characteristically different system structures starts under the action of $u = -k$ $(\sigma > 0)$ and $u = k$ $(\sigma < 0)$. This makes up the chattering about the switching manifold with a chattering magnitude proportional to the gain k. This observation brought the notion of achieving a smooth SMC by manipulating the controller gain[9].

[9]The controller gain can be made function of the sliding variable such that the gain exactly vanishes at the origin.

2.5.1 First Order Smooth Sliding Mode Control

W. Gao and his co-researchers [23] pioneered a first order smooth SMC. They proposed the gain to be a function of the sliding surface, which reduces as the surface is reached, and becomes exactly zero at the surface. This reaching law was termed as power rate reaching law and is given as:

$$u = -k\,|\sigma|^{\alpha}\,sign(\sigma) \qquad 0 < \alpha < 1.$$

In [24] a modified power rate reaching law based on the strong reachability was proposed having the following structure:

$$u = -M\sigma^{\alpha} sign(\sigma) - k\sigma^{\beta}$$

where the constants $M,\ k,\ \alpha$ and β are strictly positive. It may be observed in both cases that the robustness is lost when $\sigma \to 0$, due to the diminished gain.

In [25] an exponential reaching law has been proposed with the following structure:

$$u = -\frac{k}{N(\sigma)} sign(\sigma), \quad k > 0$$

$$N(\sigma) = \delta + (1-\delta)e^{-\alpha|\sigma|^{p}}$$

where $0 < \delta < 1$, and $p > 0$. The smoothness of the algorithm depends upon the value of δ. It may be noticed that the gain of the controller varies itself in the set $(k, \frac{k}{N(\sigma)})$. For $\delta = 1$, the algorithm reduces to a simple constant rate reaching law while for δ other than 1, the gain schedules itself as an exponential function of σ and hence state trajectories. This technique enhanced robustness and provided smoothness but the obvious requirement of very high k may cause saturation during the practical implementation. In addition, the increased computational efficiency is also a concern.

All these first order smooth sliding mode controllers were considered superior in terms of chattering suppression when compared to the approximation of sliding modes, discussed earlier. However, the research continued to present smooth sliding mode controllers which retain robustness as well. The research activities in the last decade came up with smooth as well as robust second order sliding mode algorithms.

2.5.2 Smooth Second Order Sliding Mode Control

The Smooth Second Order Sliding Mode (SSOSM) control figured out the chattering problem using the same principle adopted for smoothening the first order sliding modes[10].

In [21, 22] a concrete platform was developed for a smooth second order sliding mode control. They considered a system having relative degree one with respect to the sliding manifold σ, as follows,

$$\dot{\sigma} = f(t,x) + bu \tag{2.2}$$

where the term $f(t,x)$ is the so called drift term and b is the control channel. The authors then proposed a smooth second order SMC law for a nominal system (Eq. 2.3)[11]

$$\dot{\sigma} = u \tag{2.3}$$

as given below:

$$u = -\alpha_1 \left|\sigma\right|^{2/3} sign(\sigma) + w,$$
$$\dot{w} = -\alpha_2 \left|\sigma\right|^{1/3} sign(\sigma)$$

The robustness to unmodeled/neglected dynamics/drift-term was ensured using smooth sliding modes based robust exact differentiator as an observer. The smooth control law in combination with the disturbance observer provided robustness as well as smoothness. The resulting closed-loop dynamics were *smooth* in the sense that its discrete-time implementation did not contain high frequency components in the vicinity of the sampling rate. This fact allowed the resulted control law to be used in the outer loop, of a multi-loop system.

[10] The principle is to make the gain of the controller dependent upon the sliding surface and hence the state trajectories.

[11] The assumption was that there is no drift term.

In 2007, Shtessel *et al.* [26] proposed a SSOSM control based on the STA, for Missile guidance application. The Smooth STA (SSTA) was robustified using a first order disturbance observer to estimate the drift terms. Like wise the algorithm was devised once again for a nominal system (no disturbances and/or uncertainties). This algorithm provided a hit-to-kill accuracy in the missile guidance application. The SSOSM controllers are also extended to relative degree two systems. In 2010, Iqbal *et al.* [27] proposed a SSOSM control based on RTA, for relative degree two (with respect to the switching manifold) systems, with a second order robust exact observer in the loop. The algorithm is applied to a benchmark DC-motor speed control problem which elaborated the effectiveness (smoothness of the control signal) of this algorithm. The stability analysis and finite time convergence of these algorithms have been carried out using homogeneity approach with arbitrary gains.

In 2010, Zavala *et al.* [28] extended the well known Second-Order Sliding Mode Observer (SOSMO) of Davila *et al.* [29]. The pronounced features of the modified observer included the smoothness of the injection term and the uniformity of the algorithm with respect to the initial conditions. One other distinguishing feature of this article was the proposed Lyapunov based stability analysis, which parametrized the algorithm.

2.6 Stability Analysis And Finite Time Convergence Of SMC

Some of the most pronounced considerations of proposing an SMC algorithm are:

- Proving the stability, in the presence of uncertainties/disturbances (vanishing and non-vanishing), mathematically as well as practically, when the algorithm is brought into the closed loop with a dynamical system.

- Proving the finite time convergence of the closed loop dynamics.
- Analyzing the stability and finite time convergence in terms of the controller parameters. This is simply called ***parameterizing the algorithm***.

The last point above is of importance because a parameterized algorithm is easy to use with respect to the choice of its parameters.

Each advent in the field of SMC theory brought with itself a mathematical design for analyzing the stability and proving the finite time convergence. Initially, the Lyapunov approach was adopted for proving the existence of sliding modes in a pre-defined switching manifold. The existence of sliding modes, was equivalent to stability. The Lyapunov approach performed well for the conventional FOSM control. The proposal of HOSM algorithms created an ambiguity, regarding the stability investigation and substantiating that they converge in finite time, which was, *how to find (choose) a Lyapunov function*.

The course of HOSM control reported numerous tools for demonstrating the stability and finite time convergence. Among these techniques the most effective methods included geometric approach [30] and homogeneity approach [31], which are reported for the stability analysis and proving the finite time convergence of some famous HOSM algorithms like, RTA, Smooth RTA (SRTA), STA and SSTA. The geometric approach gave some insight to the dynamics offered by STA, e.g. *twisting* of the closed loop system trajectories around the origin and eventually converging to the origin, provided a solid proof of the stability while the homogeneity approach got popularity for the reason of being systematic.

These approaches (geometric and homogeneity) effectively provided the proofs for stability and finite time convergence but yet some questions were there to be answered.

- How could the convergence time be determined?

- What are the effects of different controller parameters (e.g. gain of the controller etc) on the stability of the closed-loop system?

- How were the controller parameters and the convergence time related to each other?

- How to set the controller parameters right, i.e. any analytical expressions for setting up various control parameters?

- What were the bounds on the closed loop system trajectories if the system has to be operated under the effects of some non-vanishing perturbations?

- What about proving homogeneity when the system is not nominal?

A Lyapunov function, for the robust stability analysis and demonstration of the finite time convergence of the STA, was proposed for the first time in Moreno et al. [32]. The approach parameterized the algorithm's robustness and performance properties and gave more detailed dynamical view. Moreover, the approach can readily be utilized for performance improvement of STA by addition of linear correction terms. In [33], a linear framework was set for the robust stability analysis of STA on the basis of the approach coined in [32]. This linear framework gave more insight of the algorithm and allowed to study it like Linear Time Invariant (LTI) systems. In Moreno et al. [34], a concrete background was laid for developing strict Lyapunov functions. The proposed approach was systematic and computationally very effective and efficient. In addition, the analytical expressions for the STA's convergence time, bounds on closed loop system trajectories and gains, were proposed for the first time.

2.7 Research Gaps

1. Since there is no reaching phase and hence no order reduction in ISMC. So any parametric variations in the underlying system, can severely degrade the

closed loop performance. This possible performance degradation due to any parametric variations can be researched for a remedy. One possible solution is to *integrate* a Linear Parameter Varying (LPV) based gain scheduling state feedback controller within the ISMC algorithm.

2. The SSTA may be parameterized, designed and proved in the presence of external disturbances[12].

3. The SSTA may be explored under the effects of some non-vanishing perturbations because perturbations of non-vanishing type has a dynamic influence of stopping the system trajectories to go to the origin. In such case, the system trajectories may only be confined to some manifold by a controller. The current work strives finding what is that manifold and how these perturbations are going to influence the performance of SSTA.

4. The integral sliding mode control can be made continuous such that it is applicable in very sensitive application e.g., a multi-loop control scheme.

2.8 Summary

The SMC, being robust, was accompanied with two inherent drawbacks such as the sensitivity to disturbances/uncertainties in reaching phase and chattering.

The ISMC, being famous for reaching phase elimination, nullified the hazard of sensitivity in reaching phase. At the same time the fact that ISMC offer no order reduction in sliding phase, created a possibility of performance degradation in the presence of varying parameters in the closed loop systems.

A world wide research on chattering reduction proposed approaches such as boundary layer, use of observers and some variants of SMC such as DSMC and HOSMC.

[12]As SSTA and many other smooth second order sliding mode controllers were devised, neglecting each and every aspect of the dynamical system except the controller.

The farmer methods reduced chattering by approximating SMC while the variants achieved chattering reduction without approximating the SMC. Among these, the HOSMC got popularity due to their easy applicability and generalized structures. The HOSMC were reported to be very sensitive to un-modeled fast dynamics and chattering may appear sooner or later in the system. The smooth SMC filled this gap by providing a continuous control action.

Initially, the Lyapunov's stability theory served as a tool for proving the stability of SMC. The advent of HOSMC brought geometric and homogeneity approaches, for the stability analysis. The stability and finite time convergence were well established with geometric and homogeneity approaches but they failed to parameterize the HOSMC. The research thus found Lyapunov functions for the HOSMC, which proved the stability and finite time convergence and also parameterized the HOSMC.

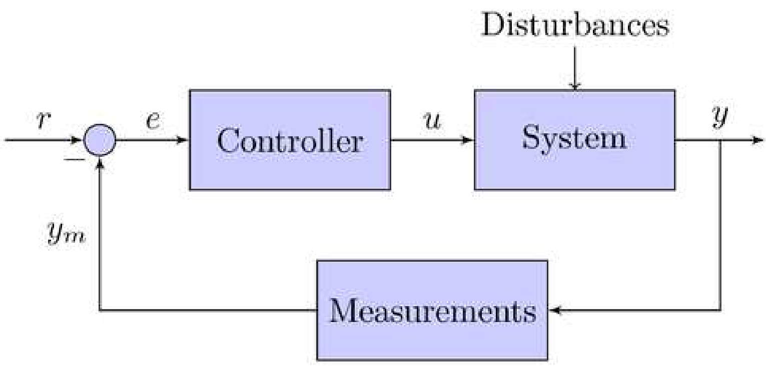
The current survey explores that the ISMC can be researched for avoiding the possible performance degradation, due to the varying parameters. In addition, the ISMC can be made continuous so that it may be applicable in many control applications. The survey also notify the need for an SSTA whose performance is parametrized in terms of its parameters.

Chapter 3

SLIDING MODE CONTROL: THEORY AND APPLICATIONS

"Start by doing what's necessary; then do what's possible; and suddenly you are doing the impossible."

Francis of Assisi.



The modern age is populated with a variety of machines and equipment to facilitate the human being. These machines include automotive, nuclear reactors, robots, computers, industries and many other machines of daily use in our houses. The sophistication of these machines is determined by various factors such as material used, strength, weight, portability, compatibility and reliability. In addition to the aforementioned factors, a well disciplined and controlled operation of these machines is also a vital consideration. These factors are researched a lot in the industry and the academia and along with many other subjects, the control system emerged as a multi-disciplinary science. The word **"control"** and hence the subject of control system is spread over a wider spectrum including mechanical, electrical, biological, chemical and social control systems. A control system is a piece of hardware or a software, designed to achieve a pre-specified goal. In other words, we can say that a control system supervises the dynamics of a component or set/s of components such that the goal, for which the control system was designed, is achieved. Thus, we can say that a control system is a setup that strives to utilize the available resources, in accordance to our demands.

One of the major concerns in analyzing and synthesizing a control system, for some physical system, is the mathematical description of that system. The mathematical models of physical systems are not always precise. Thus, a good control engineer will always look for the control system that offers a better performance and robustness in the presence of external disturbances and uncertainties/imperfection in the mathematical model.

The subject of control system deals with the analysis as well as the controller design of dynamical system, using any one of its two broad and pronounced categories, namely, the nonlinear control techniques and the linear one. The linear control techniques consider the linear approximation of the mathematical model of a physical system and are further subdivided into frequency domain (Root Locus, Bode Diagram and $H\infty$) and time domain techniques e.g., state space. The domain of linear control is equipped with the fascination of having systematic and methodical tools for analysis and design purposes but suffers from the fact that they are valid locally[1]. On the other hand, the domain of non-linear control theory utilize the non-linear mathematical models of physical systems, for both design and analysis purposes. Examples of nonlinear techniques include back stepping, Sliding Mode

[1]The linear controllers are concerned with the stability and performance of dynamical system in the vicinity of an operating point, usually an equilibrium point, around which linearization holds true.

Control (SMC) and input-output linearization etc. The nonlinear techniques has the fascination of quaranteeing global behavior of dynamical systems.

The rest of this chapter is organized as follows.

In Section 3.1 a historical background of some linear control techniques and their various aspects are given while Section 3.2 is dedicated to nonlinear control techniques. This section also gives a comparison between the linear and nonlinear control techniques. Section 3.4 describes in detail the various terminologies of sliding mode control with the help of numerical and simulation examples. In Section 3.6 and Section 3.7 an introduction to the higher order sliding modes and smooth sliding mode control, is given respectively. Section 3.8 gives the problem statements and Section 3.9 gives a short summary of the intended research work. Section 3.10 summarizes this chapter.

3.1 Linear Control Techniques

The linear control techniques utilize the linear approximations of mathematical models of physical systems and have the advantage of having generalized and systematic rules for analysis and design. However, the linear control techniques are accompanied by two major disadvantages.

1. A linear model is always an approximation of a non-linear model (neglecting some system dynamics).
2. A linear controller is valid around a specified operating point, usually some equilibrium point, which means local stability.

All the linear control techniques can be classified as classical control, modern control and Robust Control.

The classical control techniques, introduced in early 1930s, provided graphical solutions to control problems e.g. transient response adjustment (rise time (t_r), settling time (t_s), peak overshoot (M_p)) and steady state adjustment (tracking control problem). Examples of classical control include Root Locus [35], Bode plots [36] and Nyquist stability criterion [37]. All these graphical methods, predict the dynamical behavior of an open-loop transfer function when the loop around it is closed. Both Root Locus and Bode Diagram use lead compensation, lag compensation or combined lead-lag compensation to get a desired dynamic response. On the other hand the Nyquist stability criterion, which is actually a polar representation of the bode phase and magnitude plots, has the advantage of handling the time delays and non-minimum phase systems. However these graphical techniques have certain limitations like:

- They are limited to linear time invariant (LTI) systems.
- They can handle Single-Input-Single-Output (SISO) systems only.
- It is difficult to work them out for complex systems.
- They can not handle irrational functions such as delays (delays need to be approximated for both bode and root locus e.g. Pade approximation).

The modern control techniques utilize state space representation to deal with control problems. Despite the transfer function representation, the state space method develops a relation between the inputs and the outputs of a dynamical system via the first order Ordinary Differential Equations (ODE). The state space method has the following superiority merits over classical control (frequency domain) methodologies.

- It can handle Multi-Input-Multi-Output (MIMO) systems.
- The dynamical analysis and controller synthesis for complex systems becomes easier with state space approach.

- In transfer function representation the stability is termed as Bounded-Input-Bounded-Output (BIBO) and a rare but possible problem of pole-zero cancellation can cause serious problems when the control system is implemented physically while in state space representation the stability is coined as *internal stability* and there is no question of pole-zero cancellation.

There are also some other methods of linear control e.g. Proportional-Integral-Derivative (PID) control [38], H_∞ control [39] and optimal control [40]. PID control is the most widely used linear control technique which adjusts the control input (u) to the system, in accordance with the error between the actual and desired output of the system. Optimal control strives for finding the best possible controller for the accomplishment of a control problem. The H_∞ control got popularity due to the inclusion of parameters for robustness, in the controller design process. A more detailed study of linear control theory can be found in [38, 39, 41] and [42].

3.2 Non-Linear Control Techniques

The subject of non-linear control deals with the analysis and controller design for non-linear plants directly (no linear approximation is required).

3.2.1 Non-Linear Versus Linear Control

1. The fact that in non-linear control the non-linear model is used directly, gives the major advantage of non-linear control theory, because it eliminates the imprecision caused by approximating a non-linear model.

2. The stability and desired dynamic response by a linear controller is guaranteed locally in the vicinity of a pre-specified operating point, beyond which a linear controller may act poorly or cause instability. On the other hand the

non-linear control gives, in general, a wider range for controller to operate properly (global behavior) [43].

3. The hard nonlinearities (backlash, hysteresis, saturation and coulomb friction etc) [43, 44], often encountered in control engineering practices, are non-linearizable and so linear control theory can not devise a controller for these. On the other hand the non-linear control deals directly with these hard nonlinearities.

4. The nonlinear control theory does not possess much generalized rules for dynamical analysis and controller synthesis while, the linear control theory is based on generalized and well established mathematical tools for analysis and controller design.

5. Most of the linear controller synthesis is normally carried out with the assumption that the parameters of the model are well known (neglects parametric uncertainties) while the non-linear control may provide robustness against parametric uncertainties.

6. Analyzing the dynamical behavior of non-linear systems is more difficult than the linear systems.

There are various methods for analyzing and designing the non-linear control systems but none of them can be claimed to be universal [43] because, a direct solution to a non-linear differential equations is usually impossible and frequency domain transformations don't apply [43, 44]. These methods include, phase plane analysis, Lyapunov analysis, describing function and center manifold theorem etc. The two most commonly used methods for the analysis of non-linear systems are given below.

- Phase Plane Analysis [43]: It is a graphical method of finding the solution of second order non-linear differential equations. It has the advantage that it provides physical insight of the system motion (nonlinearities in the system) and a disadvantage that it is limited only to second order differential equations.

- Lyapunov Analysis: Lyapunov theory [43, 44] is the most powerful tool for the analysis of non-linear systems. The Lyapunov analysis is used in two ways, namely *Direct Method* and *Indirect Method*. The indirect method gives the stability of a non-linear system in the vicinity of an operating point via the use of linear control techniques. The direct method utilizes the energy concept for claiming the stability. A disadvantage is that finding an *energy like* function for a specific system is a difficult task. Moreover, as given in Slotine et. al. [43] **"if Lyapunov does not prove the stability of a system it does not mean that the system is unstable rather Lyapunov function fails to prove the stability".**

The design of a control system has two pronounced aspects i.e., stabilization (regulation) and tracking [43]. In addition to many other techniques for achieving these two objectives, the linearization of nonlinear system is also used as tool such that the design process is benefited from the luxury of linear controller design techniques. The linearization about an operating point, which results in an LTI representation of the nonlinear system, suffered from not covering the whole operation range. Thus the concept of global linearization brought the idea of Linear Parameter Varying (LPV) systems and hence LPV based gain scheduling. The LPV based gain scheduling will be discussed in fair detail in the subsequent section.

3.3 LPV Based Gain Scheduling

An LPV based gain scheduling is a linear controller which monitors the operating conditions of the physical system and continuously changes its parameters [45, 46].

The design of such global linear controller may involve the following steps.

- **Construction of an LPV model/system/plant:** An LPV system is a special type of representation of the nonlinear systems in which the system appears "linear" in states but the coefficients are dependent of exogenous time varying parameters which can be measured or estimated. The ideal performance or less conservatism of an LPV based gain scheduling controller requires an LPV system as close to the nonlinear system as possible. However, the LPV representation of a nonlinear system is not unique and the level of performance is strictly dependent upon the LPV description. The simplest but not trivial way of obtaining an LPV description is to hide the nonlinearity in the parameter. Consider the following simple example.

  Example 3.1. *Consider the nonlinear system*

  $$\dot{x} = -\sin(x)$$

  can be represented as an LPV system by choosing $\alpha = \sin(x)/x$*, as a scheduling parameter.*

  $$\dot{x} = -\alpha x$$

- **Controller Synthesis:** After an acceptable LPV model is achieved, the controller can be synthesized similar to the LTI counterpart e.g., using H_∞, state feedback or any other generalized technique.

- **Controller Scheduling:** The scheduling scheme for the synthesized controller i.e., computation of the gain scheduling, is performed in this step.

- **Performance Assessment:** The synthesized controller with the chosen scheduling scheme is simulated and tested on the originol nonlinear system to see whether it is acceptable or the steps are needed to be revised.

The most commonly used methods to achieve these two objectives, in nonlinear control theory, are the feedback linearization and the Sliding Mode Control (SMC).

Feedback linearization suffers from model imprecision and implementation cost, as extra sensors are required to measure all the states of the system, while the *Sliding Mode Control* got popularity due to its robustness against model uncertainties and lower implementation cost than the feedback linearization. The theory of *Sliding Mode Control* is studied in comprehensive details in the upcoming sections.

3.4 Sliding Mode Control

The term variable structure control was introduced by S. V. Emelyanov [2]. Later on in [3], the term **sliding modes** was coined for the first time and the theory of SMC was formally introduced in [6, 9, 10, 11].

One of the most inspiring feature of the SMC is that the controller is of discontinuous nature. The prime function of this controller is to provide necessary switching between two symptomatically different system configurations such that another type of system motion, known as the **sliding modes**, subsists in a manifold, known as the switching line or sliding surface or sliding manifold. This inspirational system spectacle outcomes in fabulous performance of the system such as, invariance with respect to parametric variations, a remarkable robustness against model imperfections and external disturbances and is applicable to both linear and nonlinear systems.

The SMC algorithm works in two phases, the reaching phase and the sliding phase [9, 14]. In reaching phase, the system dynamics are driven, by a discontinuous controller, from initial condition/s onto a prerequisite manifold, known as the switching manifold (also known by the names sliding surface, switching surface, hyperplane and switching line), while the sliding phase is said to be accomplished when the *sliding modes* are established in the sliding manifold i.e., when the system slides on the surface towards an equilibrium.

The SMC algorithm is designed in two steps. These steps include the design of a sliding surface and a discontinuous control law.

1. **Sliding Surface Design:** The sliding surface may have one of the following forms depending upon the controller design specifications.

    - Hurwitz Polynomial as Surface: The sliding surface (S) is a monic polynomial and is a linear combination of the states of the system. These surfaces are normally used for regulation (stabilization) purposes.
    - Tracking Specific Sliding Surface: The usual option, as a surface, for tracking purposes is the error e.g., $S = E = R - Y$, where E is the error, Y is the concerned output of the underlying system and R is the desired signal/trajectory/path etc which is to be tracked.

    A critical consideration in the design of a sliding surface is to ensure the existence of *stable sliding modes*. In other words, a surface must be designed such that when the reaching phase is accomplished, the system trajectories slide towards the origin. This discussion yields the following definition of ***sliding modes***.

    Definition 3.1. Sliding Modes: The system modes/poles governing the dynamic behavior of a system as it slides along the surface.

2. **Design of Reaching/Control Law:** The control law used in SMC algorithms is generally a discontinuous function of system states. This must be designed such that the reaching phase is accomplished in finite time and the system trajectories are confined to the surface afterwards. Since this control law is responsible for the accomplishment of reaching phase so it is also called reaching law or reaching control law.

The following example will explore the design technicalities of a conventional First Order Sliding Mode Control (FOSMC).

Example 3.2. *To explain the SMC design steps/procedure and its robustness, taking a simple second order nonlinear system,*

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 + x_2 + x_1 x_2 + b(u + \Delta(t)) \tag{3.1}$$

where, $-1 \leq \Delta(t) = \sin(t) \leq 1$ *is a bounded matched external disturbance and* b *is a strictly positive constant. This system is inherently unstable as can be seen from the open loop response shown in Figure 3.1 and may be explored mathematically using the Lyapunov's indirect method for equilibrium point* $(x_1, x_2) = (0, 0)$*,*

$$J = \begin{bmatrix} \frac{\partial f_1}{dx_1} & \frac{\partial f_1}{dx_2} \\ \frac{\partial f_2}{dx_1} & \frac{\partial f_2}{dx_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$$

which has the Right Half Plane (RHP) eigen values $(0.5 \pm 0.866j)$*.*

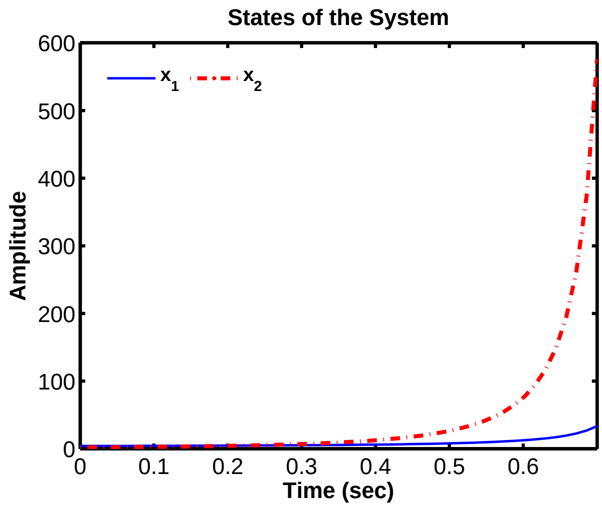


FIGURE 3.1: Open Loop Response (Response due to the initial conditions)

Hence the control objective is to stabilize this system in the presence of the given sinusoidal disturbance.

A robust sliding mode controller is devised using the following steps.

1. ***Sliding Surface Design:*** *A Hurwitz and monic linear combination of the system states is an obvious choice for sliding surface, when the objective of*

*the controller is **stabilization** of the underlying system.*

$$S = Cx_1 + x_2 \tag{3.2}$$

*Where, S denotes the sliding surface and C is a design parameter. Prior to control law design, the parameter C is designed to ensure a desired response and to make sure that the resulting **sliding modes** will be stable. Thus assuming that $S = 0$ has been accomplished by a discontinuous control law, then:*

$$x_2 = -Cx_1$$

which in tern gives,

$$\dot{x}_1 = -Cx_1 \tag{3.3}$$

Solving the above ODE gives:

$$x_1(t) = x_1(0)e^{-Ct} \tag{3.4}$$

We can conclude the following.

(a) *The surface surface (Eq. 3.2) is Hurwitz.*

(b) *The resulting sliding motion (the motion/dynamics when reaching phase has been accomplished) is represented by a reduced order dynamics.*

(c) *The sliding modes are stable (notice the decaying exponential), which ensure that the system will **slide** along the surface, towards the origin.*

(d) *The parameter C can be defined for a desired response e.g., to define a decay rate.*

2. ***Control Law Design:*** *Conventionally, the control law u, for an SMC, with a gain denoted by M, has the following form.*

$$u = u_{eq} - M sign(S) \tag{3.5}$$

Where u_{eq} is known as the equivalent control component. The u_{eq} is continuous and gives the average behavior of the system. This component reduces the *mathematical complexity in proving the stability of the system as well as reduces the high gain requirement in the discontinuous part ($Msign(S)$)*[2]*. The equivalent controller u_{eq} constitute the known system dynamics and hence it is analytically calculated for nominal system by assuming steady state ($\dot{S} = 0$).*

$$\begin{aligned}\dot{S} &= C\dot{x}_1 + \dot{x}_2 \\ 0 &= (C+1)x_2 + x_1(x_2 - 1) + bu_{eq} \\ u_{eq} &= -\frac{1}{b}\left[(C+1)x_2 + x_1(x_2 - 1)\right]\end{aligned} \tag{3.6}$$

3. ***Reachability Condition (Existence of sliding modes):*** *This step of the design process deals with the mathematical proof of the existence of sliding modes in the sliding manifold, defined earlier, with the application/help of the discontinuous controller. This can equivalently be stated that whether or not the sliding modes, $S = 0$ dynamics, are attained i.e., the reaching phase is accomplished.*

   The particular system (Eq. 3.1) in this example with the sliding surface given in Eq. 3.2 and the control/reaching law in Eq. 3.5, the reachability is said to be confirmed if the following Lyapunov function,

   $$V = \frac{1}{2}S^2,$$

   satisfy the following condition.

   $$\dot{V} = S\dot{S} \leq 0.$$

   From Equations 3.1-3.6, the following equations hold true.

   $$\begin{aligned}S\dot{S} &= bS\ (\Delta(t) - MSign(S)) \\ &\leq bS\ (1 - Msign(S)) \\ &\leq b\left[S\ - M\,|S|\right]\end{aligned}$$

[2]The chattering magnitude is directly proportional to the gain of the discontinuous controller. Hence, u_{eq} also helps reducing chattering hazards.

The above equation is strictly semi negative definite if and only if $M > 1$[3]. *This implies that the total energy in the system decays and eventually the sliding modes will be enforced (reaching phase will be accomplished).*

4. ***Sliding Phase:*** *As mentioned earlier, the sliding phase constitute a special type of system motion, enforced and maintained by the discontinuous controller. The motion in sliding phase is called special because it governed by designed system modes, called the sliding modes.*

   Eq. 3.3 shows the dynamics governing the sliding motion while Eq. 3.4 show that these dynamics are asymptotically stable.

The system in Example 3.2, with the given SMC, is simulated in MATLAB/SIMULINK R2008b with the parameters given in Table 3.1.

Entity	**Value**
M	15
C	2
b	1
Initial Condition	$\begin{bmatrix} 4 & 2 \end{bmatrix}$
Step Time	Variable

TABLE 3.1: Simulation Parameters

Figure 3.2 shows the sliding surface reached by the controller (Eq. 3.5) with the corresponding controller effort shown in Figure 3.3. It may be noticed that the surface is reached in the presence of a persistent sinusoidal external disturbance (shown as a small portion of Figure 3.2). Thus, the robustness claim is verified. Furthermore, the state trajectories under the effect of the SMC are given in Figure 3.4, which shows that the states are not only stabilized but also converge in finite time. In Figure 3.5 the phase portrait of the system states is given which elaborates the accomplishment of reaching phase and sliding phase.

[3] In general it must be ensured that $M > \eta$, where η is the upper bound of any matched external disturbance.

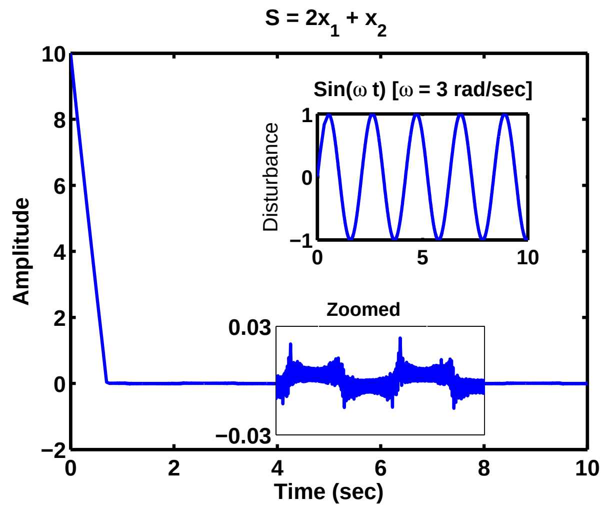


FIGURE 3.2: Sliding surface and the applied external disturbance

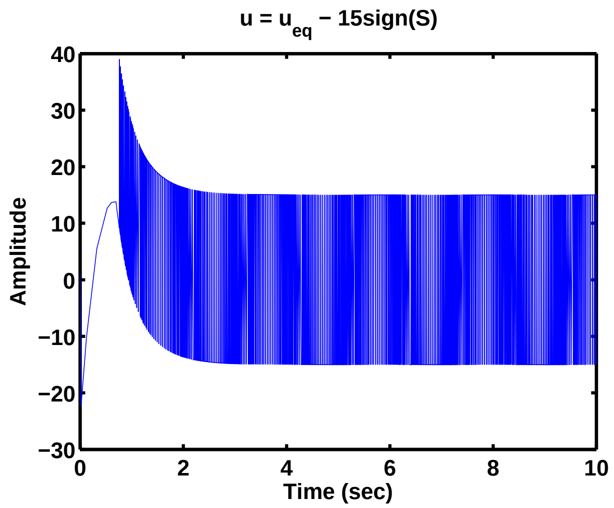


FIGURE 3.3: Controller effort

In addition to the fascinating benefits of SMC, there are also some limitations. The discontinuous nature of u, delays in mechanical systems, neglected or unknown dynamics and imperfection of hardware, results in high frequency oscillations against the sliding manifold. These high frequency oscillations are known as *chattering* [47], see the zoomed portion of Figure 3.2. These high frequency oscillations may cause wear-tear in the system. Moreover, the discontinuous controller switches with a high frequency and a fixed magnitude proportional to the gain M, see Figure 3.3. Here arises a water bed effect which is, reducing M decreases robustness

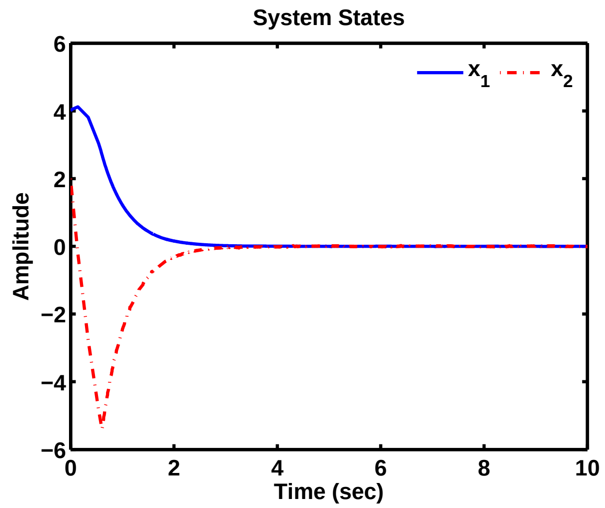


FIGURE 3.4: Closed loop state trajectories

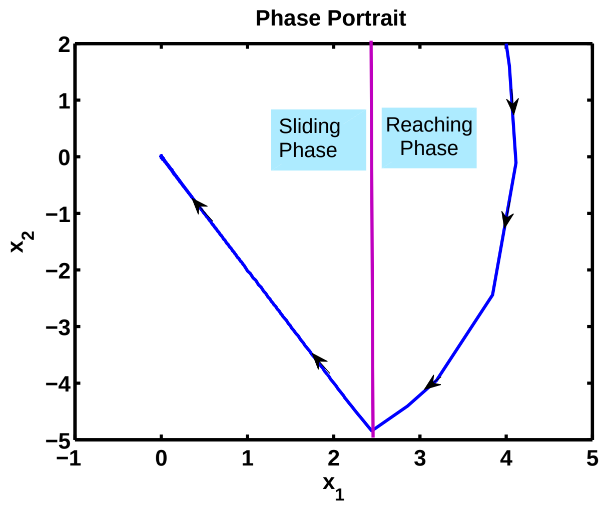


FIGURE 3.5: Phase Portrait describing the reaching and sliding phase

and increasing M causes the magnitude of the controller effort to rise. Both these situations look threatening from a control engineer's point of view and a trade-off has to be made between robustness and achievable performance. The conventional first order SMC also suffers from the fact that it is applicable only if the dynamics are relative degree one with respect to the switching manifold [48].

The pronounced efforts in the last few decades, for relaxing the relative degree requirement and suppression/elimination of the worst chattering phenomenon, added many techniques and variants in the theory of sliding mode control. One such achievement of the control theoreticians and practitioners is the Higher Order Sliding Modes (HOSM), which relaxes the relative degree constraints and minimizes the chattering.

Furthermore, some nonlinear systems show very sensitive behavior to even very small disturbances in the reaching phase. This sensitivity of a system in reaching phase of SMC may cause an undesirable result or even the instability of the system[4]. Thus it is needed to have a reaching phase free sliding mode. The cause oriented efforts of the researchers found out the Integral Sliding Mode Control (ISMC) as a remedy [8, 49, 50, 51].

The ISMC and HOSM are discussed in the coming sections.

3.5 Integral Sliding Mode Control

The Integral Sliding Mode Control (ISMC), being famous for reaching phase elimination, diminishes the hazard of possible instability and/or performance degradation due to the external disturbances, in the reaching phase. The elimination of reaching phase enhances the robustness to external disturbances but excludes the property of order reduction from the conventional SMC.

The attractive phenomenon, of reaching phase elimination, in the ISMC is attained by a special type of sliding surface, usually known as the integral manifold. The integral manifold is an algebraic sum of the conventional sliding surface (linear combination of states) and an integral term. The integral term can be termed as added dynamics and is a function of system's states and parameters[5]. The corresponding ISMC controller have a discontinuous control component, responsible for keeping the system trajectories confined to the integral manifold by rejecting

[4] As the invariance is claimed only in the sliding phase.

[5] It is these added dynamics which pulls the property of order reduction, in the sliding mode, out of the conventional SMC.

the uncertainties and disturbances of a certain class and a continuous control component, usually a state feedback controller, responsible for stabilizing the sliding mode (nominal) dynamics. A simple introduction of the ISMC is presented below.

Consider a Single-Input-Single-Output (SISO) nonlinear system with the following state space representation.

$$\dot{x}(t) = f(x,t) + B(x,t)u(t) + \xi(x,t) \tag{3.7}$$

where $x(t) \in R^n$ is the state vector, $u(t) \in R$ is the control input, $f(x,t) : \Re^n \times \Re \rightarrow \Re^n$ is a smooth vector field and $B(x,t) : \Re^n \times \Re \rightarrow \Re^n$ is a smooth input channel. In practice there are always uncertainties and external disturbances affecting the dynamical behavior of systems. These are mathematically represented by $\xi(x,t)$.

Assumption 1. *Assume that:*

1. $|B(x,t)| \neq 0$ *i.e., this matrix is full rank.*
2. *The system is controllable.*
3. *The nominal system (ideal case), with state vector* $x_{nom}(t) \in \Re^n$*, operates under the effect of state feedback control law* $u_0(x_{nom}) = -kx_{nom}$

Assumption 2. *The following assumptions are made regarding* $\xi(x,t)$*.*

1. *The uncertainties and disturbances are of matched nature i.e., the term* $\xi(x,t)$ *affects the system through input channel. This assumption provides the following mathematical equivalence.*

   $$\xi(x,t) = B(x,t)\eta \tag{3.8}$$

2. *The uncertainties and disturbances are norm bounded.*

   $$||\xi(x,t)|| \leq \delta$$

   where, δ *is a known positive constant.*

3. The term $\xi(x,t)$ vanishes at the origin i.e.,

$$\xi(0,t) = 0$$

The system operating under the effect of u_0 (the nominal system or the sliding mode dynamics) may have the following form,

$$\dot{x}_{nom} = f(x_{nom}, t) + B(x_{nom}, t)u_0 \tag{3.9}$$

where x_{nom} represent the nominal state trajectories.

The core objective of an ISMC algorithm is to enforce nominal system, the one having no uncertainties and/or disturbances, from the very beginning. Mathematically,

$$x(0) = x_{nom}(0),$$

and this retained afterwards i.e.,

$$x(t) = x_{nom}(t).$$

Therefore, the ISMC algorithm uses a linear combination of two controllers as given below.

$$u(t) = u_0(t) + u_1(t), \tag{3.10}$$

The controller $u_1(t)$, is the basic building block and is discontinuous in nature, copes with uncertainties and disturbances to achieve the above mentioned objective while the controller $u_0(t)$ is linear or nonlinear or continuous or discontinuous controller used to stabilize the nominal system.

As mentioned earlier the ISMC uses a special type of sliding surface known as the integral manifold and is given by the following equation.

$$s(x) = s_0(x) + z. \tag{3.11}$$

Where $s_0(x) = \sum_{i=1}^{n} c_i x_i$, with $c_n = 1$, is the conventional sliding surface while z is the integral term which will be explored in the subsequent paragraphs.

The time derivative of Eq. 3.11, along the dynamics of Eq. 3.7, with $u(t)$ given in Eq. 3.10, takes the form:

$$\dot{s} = \nabla s_0 \left[f(x,t) + B(x,t)u_0 + B(x,t)u_1 + \xi(x,t) \right] + \dot{z} \tag{3.12}$$

Now, selecting the integral term as a function of $f(.)$ and u_0.

$$\dot{z} = -\frac{\partial s_0(x,t)}{\partial x} \left(f(x,t) + B(x,t)u_0 \right) \tag{3.13}$$

As the ISMC is characterized by *no reaching phase* i.e., $s(x_0) = 0$, where $x_0 = x(0)$ is the initial condition, which leads to the following mathematical description for the initial conditions of the integral term.

$$z(x_0) = -s_0(x_0).$$

The above choice of the integral term, reduces the dynamics in Eq. 3.12 to the following form.

$$\dot{s} = \nabla s_0 \left[B(x,t)u_1 + \xi(x,t) \right] \tag{3.14}$$

In order to achieve the convergence condition, the discontinuous control may be selected as follows:

$$u_1(t) = u_{eq} - k\text{sign}(s),$$

where u_{eq} is the equivalent controller and can be selected using the directions given in [5] and $k > \eta$ is the controller gain. This choice of u_1, guaranteeing the convergence, leads to the sliding mode dynamics (nominal system) given in Eq. 3.9. Notice that an obvious assumption for the above analysis is that $det(\nabla\sigma_0 B(x,t) \neq 0)$.

The ISMC with the aid of integral manifold and hence no reaching phase provides robustness from the initial instant of time with acceptable performance. However, chattering can be reduced with the use of some smoothening algorithm. In addition, the possible varying parameters in the system can seriously degrade the performance of the dynamics governed by a controller u_0. These problems are addressed in Chapter 5

The following design example explains the construction of an ISMC.

Example 3.3. *Consider the nonlinear system of Example 3.2. Now the controller u is as given in Eq. 3.10. The integral manifold, as mentioned earlier, is given by the following equation,*

$$s = s_0 + z, \tag{3.15}$$

where, s_0 is as given in Eq. 3.2[6] *and z is the integral term.*

Taking the time derivative of the integral manifold (Eq. 3.15) along the trajectories of the nominal system (Eq. 3.1), i.e., $\Delta(t) = 0$.

$$\dot{s} = \dot{s}_0 + \dot{z}$$

Now taking the component u_1 as given in Eq. 3.5 and fact that $s = 0$ from the very beginning[7]*, the above equation can be equated for $\dot{z}$, considering the steady state i.e., $\dot{s} = 0$.*

$$\dot{z} = -u_0 \tag{3.16}$$

A solution of Eq. 3.16 gives the term z in Eq. 3.15. Reconsidering Eq. 3.1 in addition to the z and u_1 given above and the fact that sliding modes are invariant to external disturbances, the following double integrator system will represent the dynamics during sliding[8]*.*

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = u_0 \tag{3.17}$$

Now the continuous part of the controller u_0 may be designed using any linear control technique e.g., state feedback $u_0 = -Kx(t)$, where K is the controller gain.

The simulation result in Figure 3.6 shows the sliding surface being maintained by the ISMC algorithm without any reaching phase.

---

[6] Generally $s_0 = \sum_{i=1}^{n} C_i x_i$ with $C_n = 1$.

[7] As there is no reaching phase.

[8] Note that there is no order reduction.

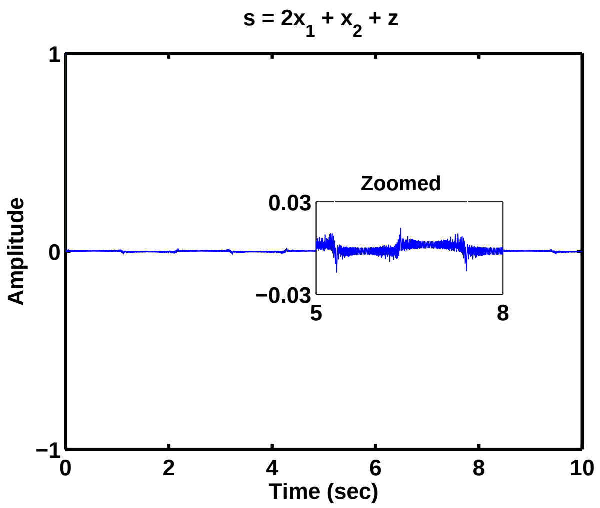


FIGURE 3.6: Integral Manifold being maintained by an ISMC

The SMC algorithms in the above two examples showing a stamped authority when it comes to robustness. However, the limitations imposed by the dangerous chattering phenomenon (see Figures 3.2, 3.3 and 3.6) and the relative degree requirement was still a matter of concern[9].

There is a rich literature of proposals claiming chattering suppression e.g. boundary layer approach, approximation of *sign function* by a *saturation function*, exponential reaching law and much more. All these proposals did well in chattering minimization but they approximated the original SMC algorithm. In the mean time the research diverted to finding techniques which could work the cause of chattering elimination without approximating the SMC. These directed efforts resulted in the introduction of HOSM control.

3.6 Higher Order Sliding Mode Control

The Higher Order Sliding Mode Control (HOSMC) [14, 17, 52] is a variant of the conventional SMC. The HOSMC relaxes the requirement of relative degree and reduces chattering.

[9]The conventional first order SMC can only be applied systems having relative degree 1 with respect to the sliding surface.

The HOSMC actually reduces the physical dimensions of the switching manifold, as the switching takes place on the intersection of switching surface (S) and its time derivatives. This leads not only to $S = 0$ but also its derivatives are forced to zero. The HOSMC algorithm which produces $S = \dot{S} = 0$ is known as the Second Order Sliding Mode (SOSM) control. Mathematically, an m^{th} order sliding mode control can be claimed if and only if the algorithm enforces the following equation.

$$S = \dot{S} = \ddot{S}......... = S^{m-1} = 0$$

The fact that an increase in sliding order actually decreases the manifold dimension, guarantee the attenuation of chattering [1]. A description of this fact can be seen in Figure 3.7, where σ is the switching line and $\dot{\sigma}$ is the total time derivative of the switching line. In short, it may be coined that there is inverse proportionality between sliding order and chattering. It may be observed that in first order sliding

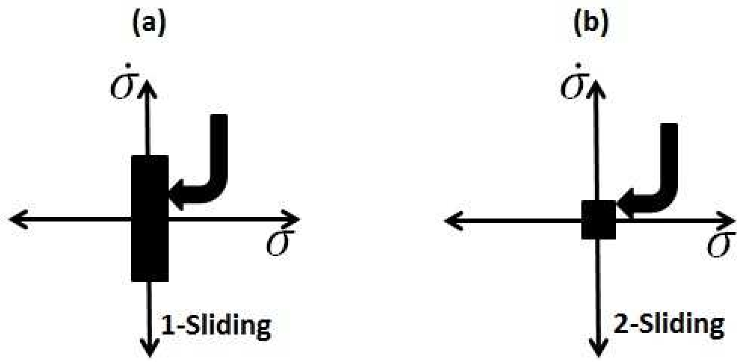


FIGURE 3.7: Description of Sliding Order and Manifold Dimension [1]

mode or conventional sliding mode, the algorithm is meant to attain $\sigma = 0$, hence the switching line is the whole $\dot{\sigma} - axis$ (large area for switching to take place). On the other hand, in second order sliding mode or 2-sliding mode, the algorithm is meant to enforce $\sigma = \dot{\sigma} = 0$ and hence the manifold dimensions are reduced.

The second order SMC ($S = \dot{S} = 0$) is the most popular amongst the researchers and is successfully implemented for many practical applications [53, 18, 54, 55, 56, 57]. The practical second order sliding mode controllers appear with different names like, Super Twisting Algorithm (STA), Real Twisting Algorithm (RTA) [14], suboptimal second order SMC [19] and a variety of smooth second order sliding mode controllers [1, 58].

The following example aids a highlight to the design of second order sliding mode control.

Example 3.4. *Reconsidering the dynamical system of Example 3.2, with the same persistent sinusoidal disturbance (*$\sin(\omega t)$*), to demonstrate the effectiveness (in terms of chattering reduction and disturbance rejection) of enforcing second order sliding modes.*

There is a rich pool of algorithms which enforces second order sliding modes. Amongst them, the most popular and widely used algorithm is the STA with the structure given in Eq. 3.18 below,

$$u = -\alpha\, |S|^{1/2}\, \mathrm{sign(S)} + \mathrm{u_1},$$
$$\dot{u}_1 = -\beta \mathrm{sign(S)}, \tag{3.18}$$

where α and β are the controller's tuning/design parameters.

The sliding surface is chosen to be that of Example 3.2 (see Eq. 3.2) and the controller parameters/gains, chosen for simulation, are $\alpha = 5$ and $\beta = 3$[10]. The simulation results for the closed loop system states, sliding surface and control effort are shown in Figures 3.8, 3.9 and 3.10 respectively. These figures, with the shown zoomed portions, reveal the superiority, in terms of chattering elimination as well as disturbance rejection, of enforcing second order sliding mode[11]. In addition to these results the phase portrait of the sliding surface and its first total time

[10] There is no systematic procedure for choosing these gains.
[11] As STA is proven to enforce second order sliding modes.

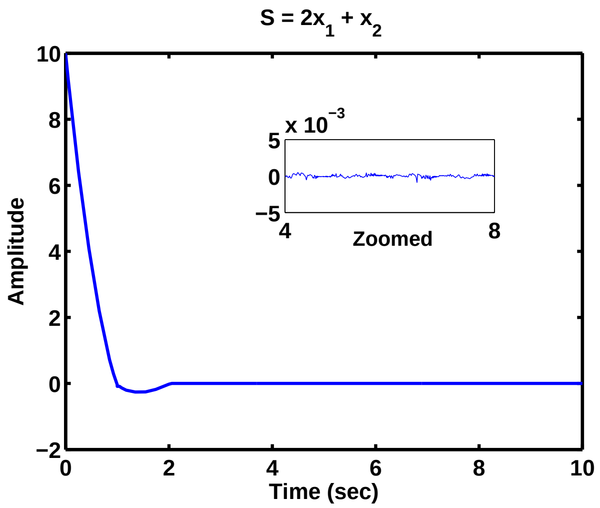


FIGURE 3.9: Sliding surface reached and maintained by STA

derivative is shown in Figure 3.11, which show the trajectories twisting around the origin $(S,\ \dot{S}) = (0,\ 0)$ *and finally reaching the origin (2-sliding mode). Hence the name* twisting controller.

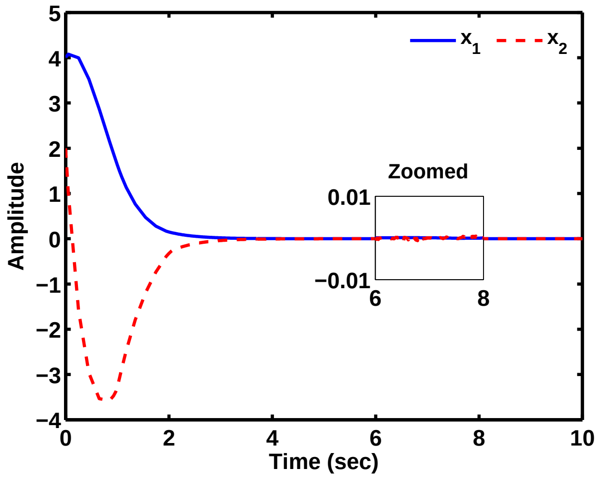


FIGURE 3.8: Closed loop state Trajectories under the effect of STA and in the presence of sinusoidal disturbance

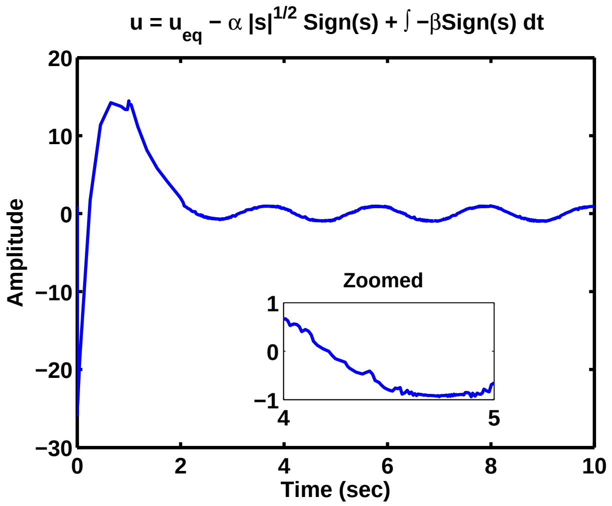


FIGURE 3.10: Controller effort produced by the STA in the presence of disturbance ($\sin(3t)$)

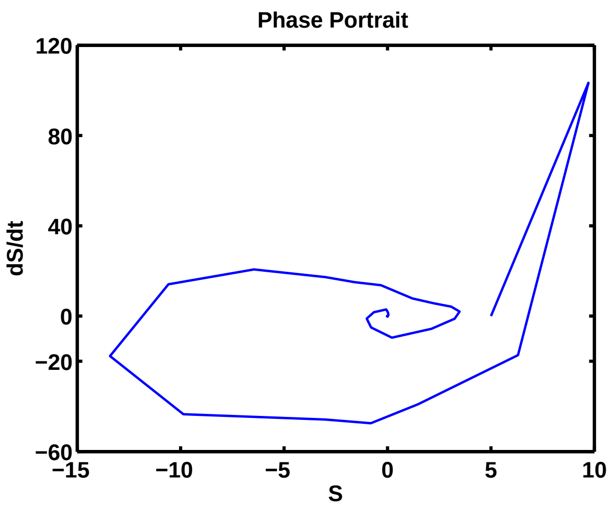


FIGURE 3.11: Phase portrait showing the twisting nature of STA and enforcement of 2-sliding mode

3.7 Smooth Sliding Mode Control

The conventional SMC had some very useful characteristics but at the same time it suffered due to high frequency chattering. The HOSM claimed to reduce the chattering effects while preserving the useful characteristics of conventional SMC. However, the HOSM control is reported sensitive to un-modeled fast dynamics [58] due to which chattering will appear sooner or later in the closed loop system.

In some very sensitive applications a complete chatter free control is required. As mentioned in Chapter 2, the second order sliding mode controllers are prone to performance degradation in such situations due their sensitivity with respect to unmodeled fast dynamics. To cope with the said performance degradation issue the smooth second order SMC frameworks were anticipated in [22, 26]. The smooth SMC algorithms characteristically produced chattering free smooth control action and hence provided guaranteed effectiveness in many sensitive applications, like the one mentioned above. The structures of Smooth STA (SSTA) and Smooth RTA (SRTA) are given in the following equations respectively.

$$\begin{aligned} u &= -\alpha \left|S\right|^{\frac{\rho-1}{\rho}} sign(S) + u_1 \\ \dot{u}_1 &= -\beta \left|S\right|^{\frac{\rho-2}{\rho}} sign(S) \end{aligned} \tag{3.19}$$

$$u = -r_1 \left|S\right|^{\frac{\rho-2}{\rho}} sign(S) - r_2 \left|S\right|^{\frac{\rho-2}{\rho-1}} sign(\dot{S}) \tag{3.20}$$

In both cases the smoothing parameter $\rho \geq 2$. Notice that when $\rho = 2$ the SSTA and SRTA reduces to the conventional STA and RTA. The gains α, β, r_1 and r_2 are strictly positive numbers.

As mentioned in previous Chapter, the robust stability analysis of each SMC algorithm is mandatory in any case. The homogeneity approach for proving the stability and finite time convergence of SSTA and SRTA, proved the stability in nominal sense. In addition, this approach do not provide any concrete evidence about the choice of the above mentioned parameters. This problem is addressed in detail in Chapter 4.

3.8 Problem Statements

1. The stability and finite time convergence of SSTA is proved using homogeneity approach which effectively determines the existence of 2-sliding mode and finite time convergence. However, the homogeneity based mathematical

proof construction does not provide any analytical solutions (equations/relations) for the gains and the convergence time of these algorithms. In addition, the effects of controller parameters (smoothing parameter and gains) on the algorithm's performance and robustness is yet to be explored.

2. The ISMC enhances robustness in the closed loop but looses the property of order reduction. The subtraction of the **order reduction property** from ISMC causes the loss of the **parameter invariance**[12]. Especially, when there are some varying parameters and/or nonlinearities in the system, the performance may degrade significantly and in worst cases it may cause instability. In addition, the ISMC do not claim any thing related to chattering suppression.

3.9 Objectives

1. To parametrize the SSTA such that the closed form expressions for selecting the gains of the controller are achieved. The parameterization is also intended to explore the robust performance and robust stability conditions in terms of the controller parameters. Such an analysis will be helpful in structural enhancement of the algorithm.

   "Robustness and Performance Parameterization of Smooth Second Order Sliding Mode Control", Vol. 13, No. 4, IJCAS 2016

2. To make the ISMC **smooth** (to eliminate chattering) for practical applications. In addition, to make a hybrid of Linear Matrix Inequality (LMI) based Linear Parameter Varying (LPV) control with a Smooth ISMC (SISMC), in order to improve performance in the presence of any time or state dependent varying parameters.

[12] As in this case the sliding mode dynamics constitute all the system dynamics.

"Smooth Integral Sliding Mode Control and LPV Approach to Uncertain Nonlinear Systems", Under Review in ISA-Transactions.

3.10 Summary

The SMC, as a nonlinear and discontinuous control scheme, is recognized due to its remarkable robustness. This algorithm tend to bring the dynamics onto a pre-defined sliding surface in two phases. The reaching phase and the sliding phase. Ideally a switching with an infinite frequency is required but practically components are not perfect and so the dangerous phenomenon called chattering appears in the system.

The HOSM control equipped with the useful properties of the conventional First Order Sliding Mode (FOSM) control, cope with the chattering by reducing the physical dimension of the sliding manifold. In addition, the HOSM algorithms allow to use the SMC for the relative degree two systems as well. The HOSM algorithms are very sensitive to un-modeled fast dynamics, due to which chattering may appear in the system sooner or later.

The thirst for a complete elimination of chattering brought the idea of continuous SMC or Smooth SMC (SSMC). The SSMC algorithms are differentiable at sampling time and hence provide a smooth/continuous control action. The stability analysis and finite time convergence of the SSMC is carried out using homogeneity approach, which does not parametrize the algorithms. The Lyapunov approach can be used as a useful tool to parametrize the SSMC algorithms.

The art of **no order reduction** in the ISMC may cause significant performance degradation if there are time or state dependent parameters and/or possible nonlinearities in the system. To cure such possible trouble the parameter invariance properties of the LMI based LPV controllers can be combined with the robustness properties of the ISMC.

Chapter 4

IMPROVEMENTS TO THE SMOOTH SUPER TWISTING ALGORITHM

"I have learn that people will forget what you said, people will forget what you did, but people will never forget how you made them feel."

Maya Angelou.

The comparison of control algorithms, is determined in the lights of properties like, simplicity of the algorithm, applicability in different circumstances, robustness against disturbances and/or model uncertainties and time domain performance.

The Sliding Mode Control (SMC), as mentioned in previous chapters, got popularity due to its remarkable robustness against uncertainties and disturbances of certain class. However, the accompanied chattering phenomenon limited their practical use. A variant of the conventional first order SMC, known as the Super Twisting Algorithm (STA), overcomes the hazard of chattering but with certain problem of sensitivity to un-modeled fast dynamics due to which chattering may appear sooner or later. This restricted the use of STA in some applications where a complete chattering free control environment was mandatory.

The Smooth Super Twisting Algorithm (SSTA) [26], a continuous version of STA and a practical second order sliding mode controller, has a simple and generalized structure with acceptable robustness and better performance. This algorithm uses a smoothening parameter, as will be discussed later in this chapter, which makes the SSTA continuous on sampling times and hence the un-modeled fast dynamics can not produce/excite chattering. However, the robust stability investigation and performance measures of the SSTA, in terms of the algorithm's parameters, is still an open field of research i.e., a proper parameterization of the algorithm is necessary in terms of design ease, performance and any possible structural improvements.

The definitions of some frequently used terms are given below.

Definition 4.1. Robustness is the property of a control algorithm, to absorb the effects of external disturbances and/or model inaccuracies [39].

Definition 4.2. Performance is the ability of a control algorithm to achieve a desired quantitative description of the parameters that constitute quality of the system behavior [39].

In this chapter, a novel robust stability analysis of the SSTA, in the closed loop, is presented. The robust stability analysis and parameterization is performed with the use of Lyapunov theory instead of homogeneity or geometric approaches, used previously. A Lyapunov function for the SSTA is constructed. The investigation of that Lyapunov function along the trajectories of the closed loop system give rise to the analytical expressions for the gains of the SSTA. The convergence time, in terms of the SSTA parameters, is also explored, which may be handy in the overall closed loop performance improvement. In short, the effect of the controller parameters on the robustness and performance of the closed loop system is explored mathematically. In addition, the challenging nonlinear process control of the Underground Coal Gasification (UCG) is controlled, using computer simulations, with the proposed analytical design for the SSTA.

The chapter is structured as follows. Section 4.1 gives the motivation of the problem. Section 4.2 presents a Lyapunov function for the nominal case and its stability and finite time convergence are mathematically elaborated. In Section 4.3, the idea of the previous section is extended to the closed loop dynamics (Eq. 4.4) perturbed by bounded matched disturbance. An analytical method, for picking the gains of the controller, is coined. In Section 4.4, the SSTA is tested/simulated in loop for maintaining the calorific value of the product gas mixture at a set point in the process control of UCG. Section 4.5 summarizes the chapter.

4.1 Problem Formulation

Consider the relative degree one sliding variable dynamics, of an output feedback plant shown in Figure 4.1, calculated along the plant trajectories,

$$\dot{\sigma} = g(t) + u, \tag{4.1}$$

where, $\sigma \in \Re$ is the sliding variable, $u \in \Re$ is the controller and $g(t) \in \Re$ is a sufficiently smooth uncertain function. In usual cases $\sigma = 0$ defines a motion on the sliding surface. However, it may be noted that the closed loop dynamics will be stabilized by u if and only if the σ-dynamics or sliding variable dynamics (Eq. 4.1) are stable.

In [26], a smooth control, modified STA, u is designed such that it drives $\sigma = \dot{\sigma} = 0$.

$$\begin{aligned} u &= -k_1|\sigma|^{\frac{\rho-1}{\rho}}\text{sign}(\sigma) + u_1 \\ \dot{u}_1 &= -k_2|\sigma|^{\frac{\rho-2}{\rho}}\text{sign}(\sigma), \end{aligned} \tag{4.2}$$

In Eq. 4.1, let $\sigma = x_1$ and $u_1 = x_2$, led us to the following σ-dynamics which may be termed as a closed loop system representing σ-dynamics.

$$\begin{aligned} \dot{x}_1 &= -k_1|x_1|^{\frac{\rho-1}{\rho}}\text{sign}(x_1) + x_2 + g(t) \\ \dot{x}_2 &= -k_2|x_1|^{\frac{\rho-2}{\rho}}\text{sign}(x_1). \end{aligned} \tag{4.3}$$

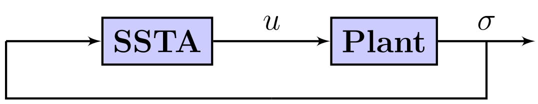


FIGURE 4.1: Output Feedback Configuration and Sliding Variable Dynamics

Where $\rho \geq 2$[1] is the smoothing parameter, $x \in \Re^2$ is the state vector, k_j, $j = 1, 2$ are the gains of the controller.

The stability of the algorithm is proved using the homogeneity approach for the nominal case ($g(t) = 0$) (see [26]). A closer look at the dynamics in Eq. 4.1 and Eq. 4.3, show that these dynamics are very sensitive to the so called drift term, $g(t)$ in this case. So the authors (see [26] and [59]) used a disturbance observer to nullify the effects of this term. However, the homogeneity lacks to deliver a concrete reasoned solutions for the controller gains and the convergence time. In addition, when the system (Eq. 4.4) is not nominal, then proving the homogeneity for the closed loop dynamics turns out to be a question. It may also be noticed that the robustness and performance of the closed loop dynamics can vary with the parameters of the disturbance observer. It will be attempted to propose a mechanism for designing gains of the controller.

Consider a modified notation, such as, $g(t) = \zeta_1(t, x)$ and adding another possible source of uncertainty $\zeta_2(t, x)$ (see [32, 34]), we re-write Eq. 4.3 as:

$$\begin{aligned} \dot{x}_1 &= -k_1|x_1|^{\frac{\rho-1}{\rho}}\,\mathrm{sign}(x_1) + x_2 + \zeta_1(t, x) \\ \dot{x}_2 &= -k_2|x_1|^{\frac{\rho-2}{\rho}}\,\mathrm{sign}(x_1) + \zeta_2(t, x) \end{aligned} \tag{4.4}$$

In this work, the controller design process (selection of the controller gains k_1, k_2 and smoothening parameter ρ) is completely parameterized using the Lyapunov approach. The parameterization is such that a selection rule is proposed for choosing the controller gains and the performance in terms of settling time, of the closed loop dyanamics, is also presented. In addition, the effect of these design parameters on the robust stability of the SSTA is explored. Moreover, the robust stability of SSTA, without requiring a disturbance observer for the estimation of the drift

[1] $\rho > 2$ is required because if it is less than 2 then the term $|x_1|^{(\rho-2)/\rho}$ will go to the denominator of the integral part ($\dot{x}_2$), which will cause infinity when $x_1 = 0$.

term, is also proven mathematically. The SSTA with the proposed design framework is then elaborated against a highly nonlinear and uncertain process control problem of UCG.

4.2 SSTA: The Nominal Case

Consider the system in Eq. 4.4 to be nominal ($\zeta_1(t,x) = \zeta_2(t,x) = 0$)[2], as given in Eq. 4.5.

$$\dot{x}_1 = -k_1|x_1|^{\frac{\rho-1}{\rho}}\text{sign}(x_1) + x_2,$$
$$\dot{x}_2 = -k_2|x_1|^{\frac{\rho-2}{\rho}}\text{sign}(x_1). \tag{4.5}$$

Taking a positive definite and radially unbounded quadratic Lyapunov function candidate [32, 34], given below:

$$V(t,x) = \xi^T P \xi, \tag{4.6}$$

where $\xi^T = [|x_1|^y sign(x_1) \quad x_2]$, $y = \frac{\rho-1}{\rho}$ and $P \in \Re^{2\times 2}$ is the solution of the Algebraic Lyapunov Equation (ALE) (Eq. 4.7) and is characterized as symmetric and positive definite,

$$A^T P + PA = -Q, \tag{4.7}$$

where $Q \in \Re^{2\times 2}$ is also a positive definite symmetric matrix.

The total time derivative of the vector term ξ, in Eq. 4.6, will evolve along the trajectories of the closed loop system (Eq. 4.5) in the following fashion:

$$\begin{aligned}\dot{\xi} &= \begin{bmatrix}\frac{\partial}{\partial t}|x_1|^y\text{sign}(x_1) & \frac{\partial}{\partial t}x_2\end{bmatrix}^T, \\ &= \begin{bmatrix}y|x_1|^{y-1}\dot{x}_1 & \dot{x}_2\end{bmatrix}^T, \\ &= \begin{bmatrix}y|x_1|^{y-1}(-k_1|x_1|^y\text{sign}(x_1) + x_2) \\ -k_2|x_1|^M\text{sign}(x_1)\end{bmatrix}.\end{aligned}$$

[2]The nominal or in other words ideal dynamic conditions is a universally good starting point in dynamic analysis and controller design.

Since $y-1 = 1/\rho$ and $M = [(\rho-2)/\rho] = y-(1/\rho)$, the above equation can further be simplified as:

$$\begin{aligned}
\dot{\xi} &= \begin{bmatrix} y|x_1|^{-1/\rho}(-k_1|x_1|^y\text{sign}(x_1)+x_2) \\ -k_2|x_1|^{-1/\rho}|x_1|^y\text{sign}(x_1) \end{bmatrix}, \\
&= |x_1|^{-1/\rho}\begin{bmatrix} y(-k_1|x_1|^y\text{sign}(x_1)+x_2) \\ -k_2|x_1|^y\text{sign}(x_1) \end{bmatrix}, \\
&= |x_1|^{-1/\rho}\begin{bmatrix} -yk_1 & y \\ -k_2 & 0 \end{bmatrix}\begin{bmatrix} |x_1|^y\text{sign}(x_1) \\ x_2 \end{bmatrix}, \\
&= |x_1|^{-1/\rho}\begin{bmatrix} -yk_1 & y \\ -k_2 & 0 \end{bmatrix}\xi, \\
&= |x_1|^{-1/\rho}A\xi,
\end{aligned}$$

where,

$$A = \begin{bmatrix} -yk_1 & y \\ -k_2 & 0 \end{bmatrix}.$$

Hence we have:

$$\begin{aligned}
\dot{\xi} &= |x_1|^{-\frac{1}{\rho}}A\xi, \\
\dot{\xi}^T &= |x_1|^{-\frac{1}{\rho}}\xi^T A^T.
\end{aligned} \tag{4.8}$$

The total time derivative of Eq. 4.6 can readily be presented based on the above derivations.

$$\dot{V}(t,x) = \dot{\xi}^T P\xi + \xi^T P\dot{\xi},$$

Put $\dot{\xi}$ and $\dot{\xi}^T$ from Eq. 4.8, we get,

$$\begin{aligned}
\dot{V}(t,x) &= |x_1|^{-\frac{1}{\rho}}\left[\xi^T A^T P\xi + |x_1|^{-\frac{1}{\rho}}\xi^T PA\xi\right], \\
&= |x_1|^{-\frac{1}{\rho}}\left[\xi^T A^T P\xi + \xi^T PA\xi\right], \\
&= |x_1|^{-\frac{1}{\rho}}\xi^T\left[A^T P + PA\right]\xi,
\end{aligned}$$

and following the definition of ALE (Eq. 4.7),

$$\dot{V}(t,x) = -|x_1|^{-\frac{1}{\rho}}\xi^T Q \xi. \tag{4.9}$$

This associate the negative definiteness of $\dot{V}(t,x)$ to the matrix Q i.e., if Q is positive definite then Eq. 4.9 is semi negative definite. In contrast, the positive definiteness of the matrix Q, by definition of the ALE (Eq. 4.7), can be claimed if and only if matrix A is Hurwitz[3]. Since the matrix P is already assumed to be positive definite and symmetric while the mathematical investigation of matrix A reveals its Hurwitz nature if and only if the gains of the SSTA (k_1 and k_2) are non-negative constants. This way the stability of the nominal system (Eq. 4.5), like a Linear Time Invariant (LTI) system, is entirely determined by the stability of the matrix A.

The following theorem formalizes the above mentioned concept for the stability of closed loop system dynamics given in Eq. 4.5.

Theorem 4.3. *If the controller gains k_1 and k_2 are non-negative then the succeeding statements are entitled for the closed loop nominal system (Eq. 4.5).*

- *A finite time stable and unique equilibrium point of the system is its origin.*
- *Second order sliding modes will be enforced in the state space of system (Eq. 4.5).*
- *Any system motion starting at an arbitrary initial conditions x_0, with any indiscriminate but symmetric and positive definite choice of matrices Q and P, will grasp the origin in time less than T_s:*

$$T_s = \frac{\rho\lambda_{max}[P]}{\lambda_{min}^{\frac{1}{\rho}}[P]\lambda_{min}(Q)} V^{1/\rho}(x_0),$$

where, $\lambda_{min}(Q)$, $\lambda_{min}[P]$ and $\lambda_{max}[P]$ are the minimum and/or maximum Eigen values of the matrices Q and P respectively.

[3]The Eigen values of A are strictly in the left half of the s-plane.

Proof. The closed loop dynamics (Eq. 4.5) is a differential inclusion ($\dot{x} \in f(x)$) and are interpreted in Filippov sense [20] because of the discontinuous right hand side. It may also be noted that $0 \in f(0)$ [60], so an equilibrium point of Eq. 4.5 is the origin. At the equilibrium $\dot{x}_1 = \dot{x}_2 = 0$, so Eq. 4.5 becomes,

$$0 = -k_1|x_1|^{\frac{\rho-1}{\rho}}\operatorname{sign}(x_1) + x_2,$$
$$0 = -k_2|x_1|^{\frac{\rho-2}{\rho}}\operatorname{sign}(x_1)$$

This equality is possible if and only if $x_1 = x_2 = 0$, which means the origin is a unique equilibrium point of the system.

The stability of the nominal system and hence the equilibrium point (origin), is directly followed from Eq. 4.9, such that

1. If $\dot{V}(t, x)$ is negative definite then the system is stable[4].
2. $\dot{V}(t, x)$ is negative definite if and only if Q is positive definite.
3. According to the definition of ALE (Eq. 4.7), Q is positive definite if A is Hurwitz (all the eigen values are strictly negative). In this regard, the characteristic equation is,

$$|\lambda I - A| = 0,$$
$$\left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} -yk_1 & y \\ -k_2 & 0 \end{bmatrix} \right| = 0,$$
$$\lambda^2 + \lambda y k_1 + y k_2 = 0.$$

   This shows that the eigen values will be strictly negative if and only if k_1 and k_2 are strictly positive ($y = \frac{\rho-1}{\rho}$ is already positive because $\rho \geq 2$). This proves the stability of the origin in terms of the controller gains.

Existence of 2-SMC: The investigation of total time derivative of the Lyapunov function (Eq. 4.9), in the lights of invariant set theorem [43], dictates the fact that

[4]Because the negative definiteness of the energy like Lyapunov function guarantee that the total energy in the system decays exponentially which is termed technically as convergence.

$\dot{V}(t, x) = 0$ on a set defined by:

$$R = \{(x_1, x_2) \in \Re^2 \quad | \quad x_1 = 0\}$$

from which the largest invariant set (say ℓ), is investigated as follows.

$$\ell = \{(x_1, x_2) \in \Re^2 \quad | \quad x_1 = x_2 = 0\}.$$

This clearly dictates that if $x_1 = x_2 = 0$ then $\dot{x}_1 = \dot{x} = 0$ (see Eq. 4.5). Thus the invariant set ℓ is a necessary and sufficient condition for the existence of second order sliding mode ($x = \dot{x} = 0$) in the state space of system in Eq. 4.5.

Finite Time Convergence: The following conclusions/claims holds true in the lights of the above discussion.

- System in Eq. 4.5 is stable.
- The only equilibrium point is the origin of the state space of Eq. 4.5.
- The gains k_1 and k_2 are strictly positive.
- The matrices P and Q are symmetric and positive definite.
- If T_s denote the convergence time of the closed loop trajectories then,

$$x(T_s) = 0 \Rightarrow V(T_s, 0) = 0.$$

The finite time convergence of the closed loop trajectories and the associated convergence time T_s is then calculated as follows.

Let $x(0) = x_0$ and $V(0, x) = V_0$, be the arbitrary initial conditions then the following inequalities are true for the quadratic Lyapunov function ($V(t, x) = \xi^T P \xi$) [43],

$$\begin{aligned} \lambda_{min}(P) \|\xi\|_2^2 \leq V(t, x) \leq \lambda_{max}(P) \|\xi\|_2^2, \\ \|\xi\|_2^2 \geq \frac{V(t, x)}{\lambda_{max}(P)}, \\ \|\xi\|_2^2 \leq \frac{V(t, x)}{\lambda_{min}(P)}, \end{aligned} \quad (4.10)$$

where, $\lambda_{max}(P)$ and $\lambda_{min}(P)$ are the maximum and minimum eigen values of matrix P respectively. Also since,

$$\|\xi\|_2 = \sqrt{|x_1| + x_2^2},$$

$$\|\xi\|_2^2 = |x_1| + x_2^2,$$

implies that

$$|x_1| \leq \|\xi\|_2^2 \leq \left[\frac{V(.)}{\lambda_{min}(P)}\right],$$

$$|x_1|^{\frac{1}{\rho}} \leq \left[\frac{V(.)}{\lambda_{min}(P)}\right]^{\frac{1}{\rho}}. \tag{4.11}$$

Note that the inequalities in Eq. 4.10 also hold true for $\xi^T Q \xi$ (see Eq. 4.9), and hence Eq. 4.9 can readily be stated as the following inequality.

$$\dot{V}(t,x) \leq -|x_1|^{-1/\rho} \lambda_{min}(Q) \|\xi\|_2^2 .$$

Now using Eq. 4.10 and Eq. 4.11, the above inequality is modified as,

$$\dot{V}(t,x) \leq -|x_1|^{-1/\rho} \lambda_{min}(Q) \frac{V(.)}{\lambda_{max}(P)}, \text{ (Using Eq. 4.10.)}$$

$$\leq -\frac{\lambda_{min}^{1/\rho}(P)}{V^{1/\rho}(.)} \lambda_{min}(Q) \frac{V(.)}{\lambda_{max}(P)}, \text{ (Using Eq. 4.11.)}$$

Simplifying the above inequality, we get,

$$\dot{V}(t,x) \leq -\delta V^y(.), \tag{4.12}$$

where,

$$\delta = \frac{\lambda_{min}^{\frac{1}{\rho}}(P) \lambda_{min}(Q)}{\lambda_{max}(P)}.$$

Eq. 4.12 proves the exponential and hence finite time stability of the closed loop dynamics (Eq. 4.5).

The use of Bihari's inequality [61] and the separation of variables approach to solving differential equations [62], for Eq. 4.12 produce the following results.

$$V(t,x) \leq \left[-\frac{\delta}{\rho} t + V_0^{\frac{1}{\rho}}\right]^{\rho},$$

At $t = T_s$ (see the bulleted points above), we have,

$$0 \leq \left[-\frac{\delta}{\rho}T_s + V_0^{\frac{1}{\rho}}\right]^{\rho},$$

$$-\frac{\delta}{\rho}T_s \geq -V_0^{\frac{1}{\rho}},$$

$$\frac{\delta}{\rho}T_s \leq V_0^{\frac{1}{\rho}},$$

which gives,

$$T_s \leq \frac{\rho}{\delta}V_0^{\frac{1}{\rho}}.$$

This completes the proof of the theorem. □

Remark 4.4. In the proof given above it may be noticed that:

- The settling time is dependent upon the controller parameters such as, the smoothening parameter ρ.
- By the virtue of Lyapunov function formulation, for the inherently discontinuous system, the stability and performance can be visualized similar to a continuous, Linear and Time Invariant (LTI) system.

4.3 SSTA with Matched Disturbances: The Non-Nominal Case

In this section, the stability of the closed loop system (Eq. 4.4) subjected to the external disturbances, which satisfies the following assumptions, is explored.

Assumption 3. *$\zeta_2(t, x)$ vanishes at the origin and $\zeta_1(t, x) = 0$. Mathematically,*

$$\begin{aligned} \zeta_1(t,x) &= 0 \quad \forall\, x \quad \textit{and} \quad t \in [0,\infty) \\ \zeta_2(t,x) &= 0 \quad \forall\, x = 0 \quad \textit{and} \quad t \in [0,\infty) \\ |\zeta_2(t,x)| &\leq L \quad \forall\, x \neq 0 \quad \textit{and} \quad t \in [0,\infty). \end{aligned} \tag{4.13}$$

With the aid of the above assumption, the robust stability analysis of the perturbed closed loop system is presented next. In addition to the robust stability analysis a selection rule, in the form of analytical expressions, is anticipated for choosing the controller gains in such a way that the assumed perturbations are compensated without requiring a disturbance observer.

Theorem 4.5. *Consider the perturbation terms in Eq. 4.4 satisfy the assumption (Eq. 4.13), then there exist, constant and positive definite symmetric matrices P and Q_p in such a way that Eq. 4.6 will be positive definite and*

$$\dot{V}(t,x) = -|x_1|^{-\frac{1}{\rho}}\xi^T Q_p \xi,$$

will be globally semi negative definite. In addition, the origin will be a global and finite time stable equilibrium point if the gains k_1 and k_2 are chosen properly, with a settling time denoted by T_{ps}.

$$T_{ps} = \frac{\rho}{\delta_p} V_0^{\frac{1}{\rho}},$$

where,

$$\delta_p = \frac{\lambda_{min}^{\frac{1}{\rho}}(P)\,\lambda_{min}(Q_p)}{\lambda max(P)}.$$

Proof. The total time derivative of Eq. 4.6, along the closed loop system trajectories Eq. 4.4 is presented as,

$$\dot{V}(t,x) = \dot{\xi}^T P \xi + \xi^T P \dot{\xi}. \tag{4.14}$$

Keeping in view the assumption (Eq. 4.13), the total time derivative of $\xi^T = [|x_1|^y sign(x_1) \quad x_2]$ is given by:

$$\dot{\xi} = \left[\frac{\partial}{\partial t}|x_1|^y \text{sign}(x_1) \quad \frac{\partial}{\partial t}x_2\right]^T,$$
$$= \left[y|x_1|^{y-1}\dot{x}_1 \quad \dot{x}_2\right]^T.$$

Now using the dynamics of the perturbed Eq. 4.4, in the above equation.

$$\dot{\xi} = \begin{bmatrix} y|x_1|^{y-1}(-k_1|x_1|^y \text{sign}(x_1) + x_2) \\ -k_2|x_1|^M \text{sign}(x_1) + \zeta_2(t,x) \end{bmatrix}.$$

Now simplifying the above equation gives,

$$\begin{aligned}
\dot{\xi} &= \begin{bmatrix} y|x_1|^{-1/\rho}(-k_1|x_1|^y \text{sign}(x_1) + x_2) \\ -k_2|x_1|^{-1/\rho}|x_1|^y \text{sign}(x_1) + \zeta_2(t,x) \end{bmatrix}, \\
&= \begin{bmatrix} y|x_1|^{-1/\rho}(-k_1|x_1|^y \text{sign}(x_1) + x_2) \\ |x_1|^{-1/\rho}(-k_2|x_1|^y \text{sign}(x_1) + |x_1|^{1/\rho}\zeta_2(t,x) \end{bmatrix}, \\
&= |x_1|^{-1/\rho} \begin{bmatrix} y(-k_1|x_1|^y \text{sign}(x_1) + x_2) \\ -k_2|x_1|^y \text{sign}(x_1) \end{bmatrix} + |x_1|^{-1/\rho} \begin{bmatrix} 0 \\ |x_1|^{1/\rho}\zeta_2(t,x) \end{bmatrix}, \\
&= |x_1|^{-1/\rho} A\xi + |x_1|^{-1/\rho}\zeta, \\
&= |x_1|^{-1/\rho}[A\xi + \zeta]. \qquad (4.15)
\end{aligned}$$

Where,

$$\zeta = \begin{bmatrix} 0 & |x_1|^{\frac{1}{\rho}} \zeta_2(t,x) \end{bmatrix}^T, \qquad (4.16)$$

and $\dot{\xi}^T$ can readily be followed from Eq. 4.15, as,

$$\dot{\xi}^T = |x_1|^{-1/\rho}[\xi^T A^T + \zeta^T]. \qquad (4.17)$$

When we put the results obtained in Eq. 4.15 and Eq. 4.17 in Eq. 4.14, we get,

$$\dot{V}(t,x) = |x_1|^{-\frac{1}{\rho}} \left[\xi^T A^T P \xi + \xi^T P A \xi + \zeta^T P \xi + \xi^T P \zeta \right]. \qquad (4.18)$$

The universal standards of linear algebra states that the transpose of a scalar is equal to that scalar. In Eq. 4.18 the terms $\zeta^T P \xi$ and $\xi^T P \zeta$ are scalars, so they are equal to their transpose. Furthermore, if we choose an arbitrary P as,

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, \qquad (4.19)$$

where $P_{12} = P_{21}$ and an M matrix as,

$$M(t,x) = \begin{bmatrix} 0 & 0 \\ n(t,x) & 0 \end{bmatrix}, \qquad (4.20)$$

where, $n(t,x) = \zeta_2(t,x)\,|x_1|^{-\frac{\rho-2}{\rho}}\,\text{sign}(x_1)$. From Eq. 4.13, one can conclude that $n(t,x)$ will be bounded. Then the following is true,

$$\zeta^T P \xi = \xi^T P M \xi,$$

and

$$(\zeta^T P\xi)^T = \xi^T P\zeta = (\xi^T PM\xi)^T = \xi^T M^T P\xi.$$

Put these in Eq. 4.18, we get:

$$\dot{V}(t,x) = |x_1|^{-\frac{1}{\rho}}\, \xi^T \left[(A+M)^T P + P(A+M)\right] \xi. \tag{4.21}$$

The upper mentioned discussion give rise to the following observations about the negative definiteness of $\dot{V}(t,x)$ (Eq. 4.21) and hence the stability of the perturbed system.

- **P is symmetric and positive definite.**

  P will be symmetric and positive definite if and only if,

  1. $P_{12} = P_{21}$.
  2. The elements of principle diagonal are positive.
  3. $det(P) > 0$.
  4. Eigen values of P are positive.

  All these are satisfied if,

$$P_{11} > 0, \quad P_{22} > 0. \tag{4.22a}$$

$$P_{11}P_{22} > P_{12}^2. \tag{4.22b}$$

Choosing $P_{11} = 1/y$ (a function of the smoothing parameter) and denote $\frac{P_{22}}{P_{12}^2} = \gamma$ then $P_{11} > 0$ ($\because \rho \geq 2 \implies y > 0$) and Eq. 4.22 becomes,

$$\begin{aligned} P_{22} > yP_{12}^2 &> 0 \\ \frac{P_{22}}{P_{12}^2} &> y \\ \gamma &> y \end{aligned} \tag{4.23}$$

Also if we let $P_{22} > P_{12}^2$, then, $\gamma > 1$, which satisfies Eq. 4.23 ($\because y < 1$).

- **$A + M$ has bounded and negative eigen values.**

  The eigen values of $A + M$ will be bounded and negative if and only if

  $$k_1 > 0,$$
  $$k_2 > n(t, x).$$

  The use of $(k_2 - L) \leq (k_2 - n(t, x)) \leq (k_2 + L)$, provide the following concluding inequalities for the eigen values of $A + M$ to be negative.

  $$k_1 > 0,$$
  $$k_2 > L. \qquad (4.24)$$

- **$(A+M)^T P + P(A+M) = -Q_p$, is satisfied, for a symmetric and positive definite matrix Q_p.**

  According to the definition of the ALE, Q_p must be symmetric and positive definite because P is a positive definite symmetric matrix and $A + M$ is Hurwitz. However, here is presented a cross check.

  Let

  $$Q_p = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix},$$

  where

  $$Q_{11} = 2[-k_1 y P_{11} + (n(.) - k_2) P_{12}],$$
  $$Q_{12} = Q_{21} = y P_{11} - y k_1 P_{12} + (n(.) - k_2) P_{22},$$
  $$Q_{22} = 2 y P_{12}. \qquad (4.25)$$

  As Q_p is already symmetric ($Q_{12} = Q_{21}$), so it will be positive definite if and only if $det(Q_p) > 0$.

  $$det(Q_p) = -4 y P_{12}[k_1 + (k_2 - n(.)) P_{12}] ...$$
  $$... - [1 - y k_1 P_{12} + (k_2 - n(.)) P_{22}]^2 > 0.$$
  $$0 < -4 y k_1 P_{12} - 4 y (k_2 - n(.)) P_{12}^2 - (1 - y k_1 P_{12})^2 ...$$
  $$... - ((k_2 - n(.)) P_{22})^2 + 2(1 - y k_1 P_{12})(k_2 - n(.)) P_{22}. \qquad (4.26)$$

Using the inequality $(k_2 - L) \leq (k_2 - n(.)) \leq (k_2 + L)$, the equation may be rephrased as:

$$0 < -4yk_1P_{12} - 4y(k_2 + L)P_{12}^2 - (1 - yk_1P_{12})^2 \ldots$$
$$\ldots - ((k_2 + L)P_{22})^2 + 2(1 - yk_1P_{12})(k_2 - L)P_{22}. \qquad (4.27)$$

Now if choose a set new variables β, γ, χ and ψ such that, $0 < \beta < 1$, $\gamma > 1$ and both satisfying $\beta\gamma > 2$ as:

$$\psi = y\,(k_2 + L)\,P_{22},\ \chi = -yk_1P_{12},\ \beta = \frac{k_2 - L}{y(k_2 + L)}. \qquad (4.28)$$

Then Eq. 4.27 can be written as:

$$\psi^2 + (\chi + 1)^2 < -4\chi - 4y\frac{\Psi}{\gamma} + 2y(\chi + 1)\Psi\beta. \qquad (4.29)$$

Since the above inequality, if changed to equality, is the equation representing the circumference of an ellipse hence it can trivially be said that this inequality in Eq. 4.29 characterizes the interior of an ellipse usually termed as sub-level trajectories sets. These sub-level sets are centered at (χ_c, ψ_c) in the plane (χ, ψ) and are branded by the parameters β, γ and ρ. This is replicated in Figure 4.2.

$$\chi_c = 1, \psi_c = y\left(\frac{\beta\gamma - 2}{\gamma}\right). \qquad (4.30)$$

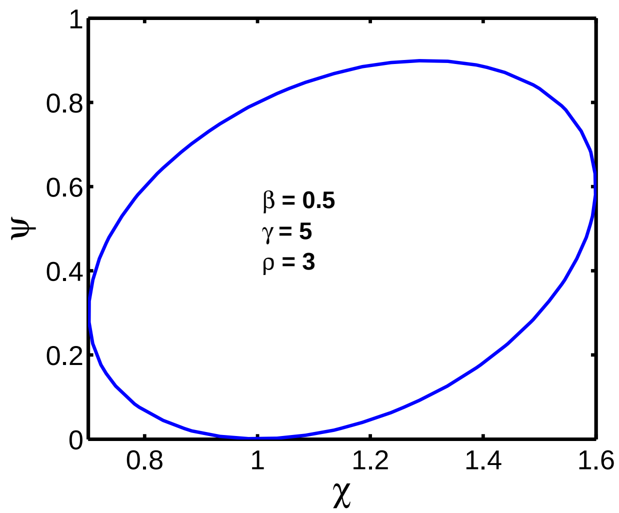


FIGURE 4.2: Ellipse describing the boundary of set (Eq. 4.29)

It may be observed from Eq. 4.30 that for $\beta\gamma > 2$ the ellipse will be centered in the first quadrant and hence Q_p will be positive definite.

The above discussion give the following set of inequalities for Eq. 4.21 to be negative definite.

$$\begin{aligned} &P_{11} > 0, \\ &P_{22} > 0 \quad \text{and} \quad P_{22} > P_{12}^2, \\ &k_1 > 0 \quad \text{and} \quad k_2 > L. \end{aligned} \tag{4.31}$$

These inequalities are satisfied with the following choices.

1. $P_{11} = 1/y$: Since $y > 0$ so $P_{11} > 0$.

2. The use Eq. 4.28, for k_1 and k_2 gives,

$$\begin{aligned} k_1 &= \frac{1}{y}\chi\sqrt{\frac{2\gamma L}{(1-\beta)\,\psi}}, \\ k_2 &= L\left[\frac{\rho\,(1+\beta)-\beta}{\rho\,(1-\beta)+\beta}\right]. \end{aligned} \tag{4.32}$$

   Clearly $k_1 > 0$ and $k_2 > L$.

With these choices Eq. 4.21 becomes,

$$\dot{V}(t,x) = -\left|x_1\right|^{-\frac{1}{\rho}}\xi^T Q_p \xi,$$

which is negative definite everywhere except at the origin. Furthermore, the finite time stability and the convergence time calculations can be performed similar to that of Theorem 1. □

In Theorem 4.5 the robustness of the closed loop dynamics with SSTA in the loop is explored, for a class of disturbances which satisfy Eq. 4.13.

Remark 4.6. A general perception, for guaranteeing the convergence of any closed loop dynamics having SMC in the loop, is that the gains of the SMC algorithm should be greater than the disturbances[5]. This requirement give rise to a trivial question: **how much greater?** An answer to this question is provided by Eq. 4.32.

The robust stability analysis of the perturbed system is presented next, for a class of disturbances/uncertainties with a non-zero and bounded $\zeta_1(t, x)$.

$$|\zeta_1(t, x)| \leq \alpha_1 + \alpha_2 \|\xi\|_2 \tag{4.33}$$

Where α_1 and α_2 are non-negative constants. It may be noticed that whether $\zeta_1(t, x)$ will vanish at the origin or not, depend upon α_1.

- $\boldsymbol{\alpha_1 = 0}$: This implies that the term $\zeta_1(t, x)$ will vanish as the origin is reached and the trajectories will eventually be confined to the origin.
- $\boldsymbol{\alpha_1 \neq 0}$: This implies the non-vanishing nature of the term $\zeta_1(t, x)$, even at the origin. Consequently, they change the equilibrium of Eq. 4.4 and hence the motions/dynamics (trajectories) will be ultimately bounded [43, 44, 61].

Theorem 4.7. *Consider the system in Eq. 4.4 with the perturbation terms $\zeta_1(t, x)$, $\zeta_2(t, x)$ and gains k_1, k_2 satisfying Equations 4.33, 4.13 and 4.32 respectively. Then the trajectories of the perturbed system (Eq. 4.4) are globally ultimately bounded by:*

$$b = \sqrt{\frac{\lambda_{max}(P)}{\lambda_{min}(P)} \frac{2\alpha_1\eta}{(1-\kappa)[\lambda_{min}(Q_p) - 2\alpha_2\eta]}},$$

for $\alpha_2 \leq \frac{\lambda_{min}(Q_p)}{2\eta}$, $\alpha_1 > 0$, $\eta \underline{\Delta} \sqrt{1 + (yP_{12})^2}$, with a symmetric and positive definite P and the non-negative constant $0 < \kappa < 1$. In addition, if $\alpha_1 = 0$, the origin will be a global and finite time stable equilibrium, with convergence time less than $\overline{T}_{1s}$,

$$\overline{T}_{1s} = \frac{\rho\lambda^y_{max}(P)}{\lambda_{min}(Q_p) - 2\alpha_2\eta} V_0^{\frac{1}{\rho}},$$

where $V_0 = V(0, x_0)$ and x_0, as mentioned above, represent the initial conditions. However, the trajectories will converge to the manifold Ω (which may contain the

[5]Normally we talk about bounded disturbances. In such cases the controller gain must be greater than the upper bound.

origin), if $\alpha_1 \neq 0$[6].

$$\Omega = \left\{ x \in \Re^2 \quad | \quad V(t,x) \leq \lambda_{max}(P)\mu^2 \right\}$$

In this case the convergence of trajectories to Ω *will take place in time which is less than* $\overline{T}_{2s}$*,*

$$\overline{T}_{2s} = \rho \frac{\lambda^y_{max}(P)}{\kappa\left(\lambda_{min}(Q_p) - 2\alpha_2\eta\right)} \left(V_0^{\frac{1}{\rho}} - \lambda^{\frac{1}{\rho}}_{max}(P)\mu^{\frac{2}{\rho}} \right),$$

where,

$$\mu \underline{\Delta} \frac{2\alpha_1\eta}{(1-\kappa)\left(\lambda_{min}(Q_p) - 2\alpha_2\eta\right)}.$$

Proof. The proof of this theorem is accomplished in two stages i.e., ($\alpha_1 = 0$ and $\alpha_1 \neq 0$).

Keeping in consideration the assumptions given in Equations 4.13 and 4.33, mentioned above, the total time derivative of ξ can be calculated as:

$$\begin{aligned}
\dot{\xi} &= \begin{bmatrix} \frac{\partial}{\partial t}|x_1|^y \text{sign}(x_1) & \frac{\partial}{\partial t} x_2 \end{bmatrix}^T, \\
&= \begin{bmatrix} y|x_1|^{y-1}\dot{x}_1 & \dot{x}_2 \end{bmatrix}^T, \\
&= \begin{bmatrix} y|x_1|^{y-1}(-k_1|x_1|^y \text{sign}(x_1) + x_2 + \zeta_1(t,x)) \\ -k_2|x_1|^M \text{sign}(x_1) + \zeta_2(t,x) \end{bmatrix}, \\
&= |x_1|^{-1/\rho} \begin{bmatrix} y(-k_1|x_1|^y \text{sign}(x_1) + x_2 + \zeta_1(t,x)) \\ -k_2|x_1|^y \text{sign}(x_1) + |x_1|^{1/\rho}\zeta_2(t,x) \end{bmatrix}, \\
&= |x_1|^{-1/\rho} \left\{ \begin{bmatrix} y(-k_1|x_1|^y \text{sign}(x_1) + x_2) \\ -k_2|x_1|^y \text{sign}(x_1) \end{bmatrix} + \begin{bmatrix} y\zeta_1(t,x) \\ |x_1|^{1/\rho}\zeta_2(t,x) \end{bmatrix} \right\}, \\
&= |x_1|^{-1/\rho} \left\{ \overbrace{\begin{bmatrix} -yk_1 & y \\ -k_2 & 0 \end{bmatrix}}^{A} \overbrace{\begin{bmatrix} |x_1|^y \text{sign}(x_1) \\ x_2 \end{bmatrix}}^{\xi} + \overbrace{\begin{bmatrix} 0 \\ |x_1|^{1/\rho}\zeta_2(t,x) \end{bmatrix}}^{\zeta} + \overbrace{\begin{bmatrix} y\zeta_1(t,x) \\ 0 \end{bmatrix}}^{B_2} \right\}, \\
&= |x_1|^{-1/\rho}\left(A\xi + \zeta + B_2\right),
\end{aligned}$$

[6]Now the trajectories will not necessarily be confined to origin rather they will be bounded by Ω.

where B_1 is arbitrary given name. Hence we have,

$$\xi = |x_1|^{-1/\rho} \left(A\xi + \zeta + B_2\right),$$
$$\xi^T = |x_1|^{-1/\rho} \left(\xi^T A^T + \zeta^T + B_2^T\right). \tag{4.34}$$

The total time derivative of Eq. 4.6 along the closed loop system trajectories Eq. 4.4 and using Eq. 4.34, is given as:

$$\dot{V}(t,x) = |x_1|^{-\frac{1}{\rho}} \left[\xi^T A^T P\xi + \xi^T P A\xi + \zeta^T P\xi + \xi^T P\zeta + B_2^T P\xi + \xi^T P B_2\right],$$

The use of results obtained in the proof of Theorem 4.5, the above equation can be rephrased as,

$$\dot{V}(t,x) = |x_1|^{-\frac{1}{\rho}} \left[-\xi Q_p \xi + B_2^T P\xi + \xi^T P B_2\right]. \tag{4.35}$$

The terms $B_2^T P\xi$ and $\xi^T P B_2$ can further be simplified as,

$$\begin{aligned}
B_2^T P\xi &= \begin{bmatrix} y\zeta_1(t,x) & 0 \end{bmatrix} \begin{bmatrix} 1/y & P_{12} \\ P_{12} & P_{22} \end{bmatrix} \begin{bmatrix} |x_1|^y \text{sign}(x_1) \\ x_2 \end{bmatrix}, \\
&= \begin{bmatrix} y\zeta_1(t,x) & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{y}|x_1|^y \text{sign}(x_1) + x_2 P_{12} \\ P_{12}|x_1|^y \text{sign}(x_1) + x_2 P_{22} \end{bmatrix}, \\
&= y\zeta_1(t,x) \left[\frac{1}{y}|x_1|^y \text{sign}(x_1) + x_2 P_{12}\right], \\
&= \zeta_1(t,x) \left[|x_1|^y \text{sign}(x_1) + y x_2 P_{12}\right], \\
&= \zeta_1(t,x)\xi^T \begin{bmatrix} 1 \\ yP_{12} \end{bmatrix}.
\end{aligned} \tag{4.36}$$

Using Eq. 4.36 and the fact that $B_2^T P\xi$ is a scalar, so it is equal to its transpose, we can further rephrase Eq. 4.35 as,

$$\begin{aligned}
\dot{V}(t,x) &= -|x_1|^{-\frac{1}{\rho}} \left[\xi^T Q_p \xi - 2B_2^T P\xi\right], \\
&= -|x_1|^{-\frac{1}{\rho}} \left[\xi^T Q_p \xi - 2\zeta_1 \xi^T \begin{bmatrix} 1 \\ yP_{12} \end{bmatrix}\right].
\end{aligned} \tag{4.37}$$

The use of inequality,

$$\zeta_1 \xi^T \begin{bmatrix} 1 \\ yP_{12} \end{bmatrix} \leq \zeta_1 \|\xi\|_2 \eta \leq (\alpha_1 + \alpha_2 \|\xi\|_2) \eta \|\xi\|_2,$$

which can directly be followed from Eq. 4.33, transforms Eq. 4.37 to the following form.

$$\dot{V}(t,x) \leq -|x_1|^{-\frac{1}{\rho}} \left[\xi^T Q_p \xi - 2(\alpha_1 + \alpha_2 \|\xi\|_2)\eta \|\xi\|_2\right].$$

Now using Eq. 4.10 with the fact that Q_p is symmetric and positive definite, we get,

$$\begin{aligned}
\dot{V}(t,x) &\leq -|x_1|^{-\frac{1}{\rho}} \left[\lambda_{min} Q_p \|\xi\|_2^2 - 2(\alpha_1 + \alpha_2 \|\xi\|_2)\eta \|\xi\|_2\right]. \\
&\leq -|x_1|^{-\frac{1}{\rho}} \left[g \|\xi\|_2^2 - 2\alpha_1 \eta \|\xi\|_2\right], \\
&\leq -\vartheta \left[g \|\xi\|_2 - 2\alpha_1 \eta\right], \qquad (4.38)
\end{aligned}$$

where,

$$g = (\lambda_{min}(Q_p) - 2\alpha_2 \eta),$$

and

$$\vartheta = \sqrt{\frac{V^M(.)}{\lambda_{max}^M(P)}}$$

If we let $\|\xi\|_2 = \kappa \|\xi\|_2 + (1-\kappa)\|\xi\|_2$, then Eq. 4.38 can be represented as:

$$\dot{V}(t,x) \leq -\vartheta \left[g\left(\kappa \|\xi\|_2 + (1-\kappa)\|\xi\|_2\right) - 2\alpha_1 \eta\right], \qquad (4.39)$$

Case 1. $\alpha_1 = 0$

In this case $\dot{V}(t,x)$ (Eq. 4.39), with the use of Eq. 4.10, takes the following mathematical representation.

$$\begin{aligned}
\dot{V}(t,x) &\leq -\vartheta g \sqrt{\frac{V(.)}{\lambda_{max} P}} \\
&\leq -Z V^y(.), \\
Z &= \frac{\lambda_{min}(Q_p) - 2\alpha_2 \eta}{\lambda_{max}^y(P)}.
\end{aligned}$$

This will be negative definite if and only if $\lambda_{min}(Q_p) \geq 2\alpha_2\eta$ and with this condition the differential inequality shows the usual exponential convergence properties. Moreover, the convergence time $\overline{T}_{1s}$ can easily be determined using the procedure in the proof of Theorem 4.3.

$$\overline{T}_{1s} = \rho \frac{\lambda_{max}^y(P)}{\lambda_{min}(Q_p) - 2\alpha_2 \eta} V_0^{\frac{1}{\rho}}.$$

Case 2. $\alpha_1 \neq 0$

In this case for $\dot{V}(.)$ to be negative definite we need,

$$2\alpha_2\eta \leq \lambda_{min}(Q_p),$$

$$2\alpha_1\eta \leq g(1-\kappa)\left\|\xi\right\|_2,$$

or

$$\left\|\xi\right\|_2 \geq \frac{2\alpha_1\eta}{g(1-\kappa)} = \mu. \tag{4.40}$$

Now using the identity in Eq. 4.10 and Eq. 4.40

$$\sqrt{\frac{V(.)}{\lambda_{max}(P)}} \geq \mu$$

$$V(.) \geq \lambda_{max}(P)\mu^2. \tag{4.41}$$

The inequality in Eq. 4.41 justifies the manifold Ω, described in Figure 4.3.

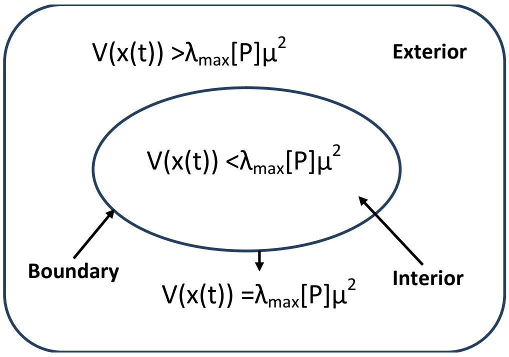


FIGURE 4.3: Description of the Manifold (Ω)

Evaluation of Eq. 4.39 at the boundary of the manifold Ω ($\left\|\xi\right\|_2 = \mu$), gives:

$$\begin{aligned}\dot{V}(t,x) &\leq -\vartheta\left[g\kappa\left\|\xi\right\|_2 + g(1-\kappa)\left\|\xi\right\|_2 - 2\alpha_1\eta\right], \\ &\leq -\vartheta\left[g\kappa\left\|\xi\right\|_2 + g(1-\kappa)\mu - 2\alpha_1\eta\right], \\ &\leq -\vartheta\left[g\kappa\left\|\xi\right\|_2 + g(1-\kappa)\frac{2\alpha_1\eta}{g(1-\kappa)} - 2\alpha_1\eta\right], \\ &\leq -\vartheta\left[g\kappa\left\|\xi\right\|_2\right], \\ &\leq -\left(\lambda_{min}(Q_p) - 2\alpha_2\eta\right)\kappa\frac{V^y(x)}{\lambda^y_{max}(P)}. \end{aligned} \tag{4.42}$$

In determining/extracting an expression for $\overline{T}_{2s}$ from the above equation, it is important to note that in this case the convergence/settling time does not mean convergence to the origin i.e., $V(\overline{T}_{2s}, x) \neq 0$[7]. The trajectories will rather converge to a manifold Ω i.e., $V(\overline{T}_{2s}, x) \to \Omega$[8]. Keeping in consideration the above guidelines, the expression for $\overline{T}_{2s}$ is worked out following the procedure in the proof of Theorem 1. □

Remark 4.8. The stability of dynamical systems subjected to non-vanishing perturbations is termed as uniform ultimate boundedness and bounds will be dependent upon (will be a function of) the terms which define that non-vanishing perturbation.

Remark 4.9. Once again it may be noticed, from the above proof, that the stability bounds and convergence/reaching/settling time are presented as functions of the controller parameters.

The upcoming sections explore the effectiveness of the SSTA, using the suggested analytical representations for its gains, when it is brought into the loop for the process control of UCG. The SSTA is meant to uphold a maximum calorific value of the product gases.

4.4 Control of the process of Underground Coal Gasification

The process of Underground Coal Gasification (UCG), as the name suggests, consumes the coal, way below the earth surface, for useful energy production. A visual of the process is depicted in Figure 4.4.

The UCG reactor is formed by boring two application specific wells, one for injection of air to initiate combustion called the injection well and a production well

[7]Because origin is not any more the equilibrium point.
[8]$\overline{T}_{2s}$ indicate the time taken by system trajectories to reach the manifold Ω.

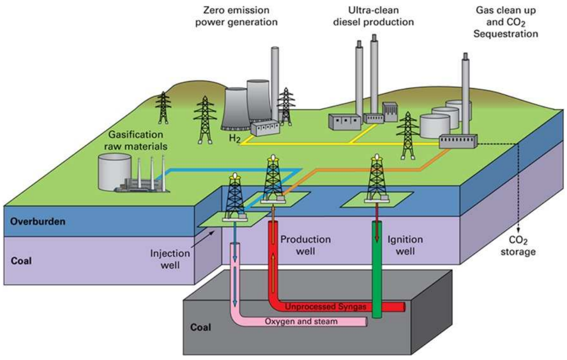


FIGURE 4.4: Underground Coal Gasification Reactor

used for collecting the product gases, from the earth surface to the coal mines. A link, either pneumatic or by horizontal drilling, between the injection well and the production well completes the reactor.

The process of UCG is initiated by igniting the coal through heaters lowered already into the coal reactor and fresh air injected through the injection well. The ignited coal then goes through the pyrolysis reaction to produce *char*. The surplus oxidants (air/oxygen) and steam (H_2O), injected through injection well, goes through a chemical reaction with *char*, called the gasification reaction, resulting in production of the all important and useful synthesis gas[9]. The synthesis gas or syn gas can be utilized as a chemical feedstock [63, 64]. It can also be used as fuel for combined cycle turbine's (CCT) operation, to produce electricity (see Figure 4.5).

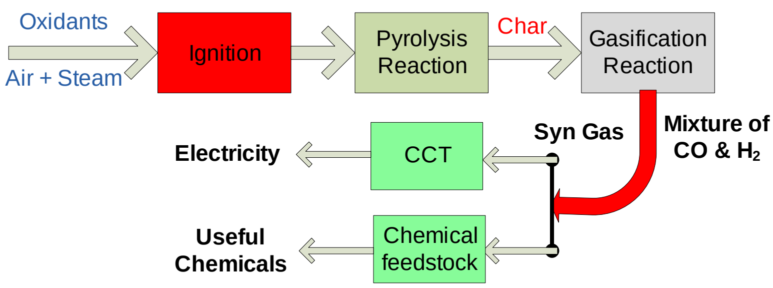


FIGURE 4.5: A Snapshot of the UCG process

[9]A mixture of carbon mono oxide (CO) and hydrogen (H_2).

The UCG reactor is modeled mathematically, including the two species/phases: gas phase/specie and solid phase/specie. A further subdivision is that the solid phase constitute two species namely coal and char, where as the gas phase is comprised of eight gases including Hydrogen (H_2), Carbon Dioxide (CO_2), Carbon Monoxide (CO), Steam (H_2O), Nitrogen (N_2), Methane (CH_4), Oxygen (O_2) and higher hydrocarbons (Tar). In [65], a one dimensional time domain mathematical representation of the UCG process is presented based on the early research (see [66] and [67]). A generalized but descriptive state space representation of the UCG process is given in Eq. 4.43.

$$
\begin{aligned}
\dot{x}_1 &= M_{coal} \sum_{j=1}^{3} a_{coal,j} r_j, \\
\dot{x}_2 &= M_{char} \sum_{j=1}^{3} a_{char,j} r_j, \\
\dot{x}_3 &= \frac{1}{C_s} \left[ht \left(T - x_3 \right) - H_s \right], \\
\dot{x}_i &= \sum_{j=1}^{3} a_{gas,j} r_j - \beta x_i, \quad where\ i = 4, 5, 6, 7, 8, \\
\dot{x}_9 &= \sum_{j=1}^{3} a_{H_2O,j} r_j - \beta x_9 + \frac{a}{L} u + \frac{\zeta}{L}, \\
\dot{x}_{10} &= \sum_{j=1}^{3} a_{O_2,j} r_j - \beta x_{10} + \frac{b}{L} u, \\
\dot{x}_{11} &= -\beta x_{11} + \frac{c}{L} u, \\
h &= mf_{CO} H_a + mf_{H_2} H_b + mf_{CH_4} H_c,
\end{aligned} \tag{4.43}
$$

where $u \in \Re$ is the input of the UCG reactor and represent the flow rate of the inlet gases (a mixture of O_2, H_2O and N_2) while h is output of the UCG process and represent the calorific value of the gases collected at the production well.

The inflow of water from the contiguous aquifers disturb the gasification reaction as they bring down the temperature of the reactor. Hence, this is treated as an

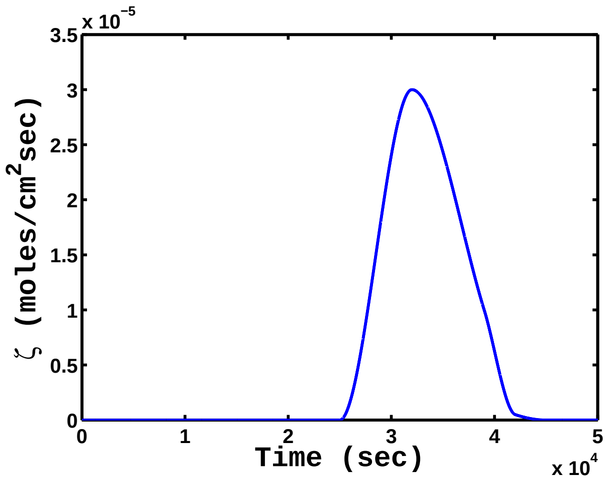


FIGURE 4.6: The profile of water influx w.r.t time

input disturbance and is denoted by $\zeta(t, x) \in \Re$. A profile of water influx from the surrounding aquifers, having an upper bound $l = 3 \times 10^{-5}$ $moles/cm^2Sec.$ is presented in Figure 4.6. The description of states and other parameters of Eq. 4.43 is given in Table 4.1 and Table 4.2 respectively.

Name	Description	Unit
x_1	Coal Density	g/cm^3
x_2	Char Density	g/cm^3
x_3	Solid Temperature	K
x_4	Concentration of CO	$Moles/cm^3$
x_5	Concentration of CO_2	$Moles/cm^3$
x_6	Concentration of H_2	$Moles/cm^3$
x_7	Concentration of CH_4	$Moles/cm^3$
x_8	Concentration of Tar	$Moles/cm^3$
x_9	Concentration of H_2O	$Moles/cm^3$
x_{10}	Concentration of O_2	$Moles/cm^3$
x_{11}	Concentration of N_2	$Moles/cm^3$

TABLE 4.1: Description of states

Name	Description	Unit
M_c	Molecular Weight of c Specie where c Represents Coal or Char	$g/mole$
$a_{i.j}$	Stoichiometric Coefficient of i^{th} Specie in j^{th} Chemical Reaction	-
r_j	Reaction Rates ($j = 1, 2, 3$) $r_1 = 5\frac{\rho_{coal}}{M_{coal}} \exp\left(\frac{-6039}{T_s}\right)$ $r_2 = \frac{1}{\frac{1}{r_{c_2}} + \frac{1}{r_{m_2}}}$ $r_3 = \frac{1}{\frac{1}{r_{c_3}} + \frac{1}{k_y m_{H_2O}}}$	-
C_s	Total Solid Phase Heat Capacity	$Cal/K.cm^3$
ht	Convective Heat Transfer Coefficient	$Cal/sec.K.cm^3$
T	Ignition Temperature	K
H_s	Solid Phase Heat Source	$Cal/Sec.cm^3$
L	Length of the Reactor	cm
mf_i	Mole Fraction of i^{th} Gas $\left(mf_i = \frac{x_i}{\sum_{i=4}^{11} x_i}\right)$	-
H_i	Heat of Combustion of i^{th} Gas	$KJ/Mole$
u	Flow Rate of Inlet Gases	$Moles/cm^2Sec$
h	Calorific Value of the Product Gas	$KJ/Mole$

TABLE 4.2: Description of parameters

4.4.1 Control Problem

The input of the UCG process is the flow rate (moles) of the injected gas mixture and hence the process is very sensitive to these. Figure 4.7 depicts the response of the UCG process to a constant input (constant flow rate of oxidants at the injection side) ($u = 2 \times 10^{-4} moles/cm^2 sec$). It may be noticed that the calorific value of the gases at the production side attains a maximum value of $113.831 KJ/mol$ in approximately $20,000$ *secs* and then starts declining thereafter. A logical reason for this decline is the leftover moles of the injected gas mixture. Thus, an automatic control system is required to manipulate the flow rate of injected gas mixture such that the decline in calorific value of the product gas is avoided.

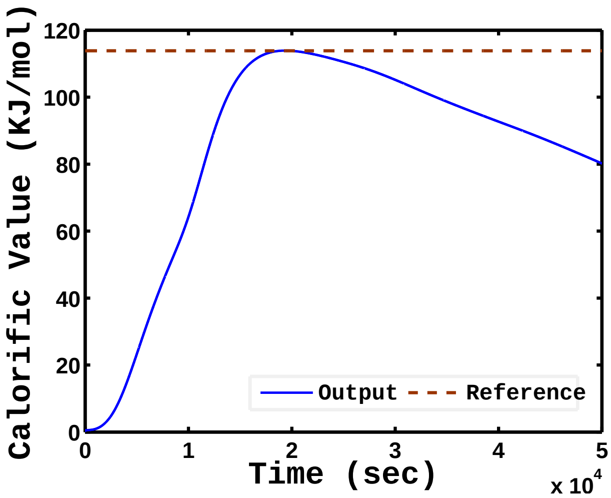


FIGURE 4.7: Open loop response of the system w.r.t time

In addition, there is also a limitation on the flow rate of injected gases, imposed by hardware (compressors). Thus, the designed automatic control must perform manipulation of flow rate u such that it stays well within the limitations presented in Eq. 4.44.

$$0 < u \leq 3 \times 10^{-4}. \tag{4.44}$$

4.4.2 Simulation Results

According to the results and findings in Section 4.3, if the parameter β is chosen to be 0.8 and $\gamma = 3$ then $\beta\gamma > 2$ and hence the center of the ellipse will be in the first quadrant i.e., $\chi_c = 1$ and $\psi_c = 0.09$ (See Eq. 4.30). The smoothening parameter ρ is selected as the smallest possible value i.e., $\rho = 3$, such that smoothening is provided and the closed loop dynamics are faster. With these values the controller gains, at ellipse center, are calculated as $k_1 = 0.15$ and $k_2 = 10*-5$ using Eq. 4.32. The simulations are then performed with these controller parameters, summarized in Table 4.3.

Parameter	**Value**
β	0.80
ρ	3
γ	3
χ_c	1
ψ_c	0.09
k_1	0.15
k_2	10^{-4}

TABLE 4.3: Simulation parameters

Notice that χ and ψ are chosen to be the center points of the ellipse (Eq. 4.29).

The computer simulations are performed with MATLAB/SIMULINK for the SSTA with the above parameters and are compared with the simulation results of a conventional First Order Sliding Mode (FOSM) control.

First of all a constant flow rate of the inlet gas mixture ($u = 2{\times}10^{-4} moles/cm^2 sec$) is injected till $20,000$ *secs*. The constant flow rate builds up sufficient calorific value of the product gases and helps eradication of the actuator (compressor) saturation when the loop is closed with SSTA or FOSM after $20,000$ *sec*.

Figure 4.8(a) demonstrate the calorific value of the product gas mixture collected at the production side for both the SSTA and FOSM. It may be observed that the desired calorific value ($113.831 KJ/mol$) of the product gas mixture is maintained by both SSTA and FOSM. However, a closer look of both (see the zoomed images in Figures 4.8(b) and 4.8(c)) reveal the robustness and smoothness of SSTA. In Figure 4.8(c) it may be noticed that the performance of FOSM is degraded by the input disturbance (water influx Figure 4.6).

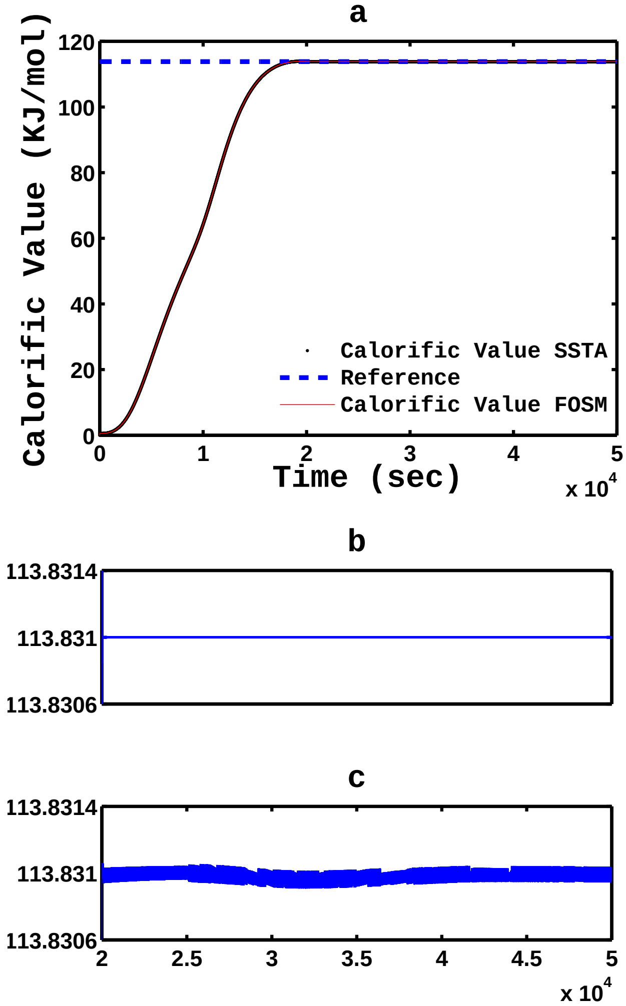


FIGURE 4.8: (a) Calorific value maintained by SSTA and FOSM controller. (b) Zoomed view of calorific value attained by SSTA. (c) Zoomed view of calorific value attained by FOSM controller

Figures 4.9(a) and 4.9(b) show the profile of the flow rate of inlet gas mixture at the injection well (control input) governed by FOSM control and SSTA respectively. These Figures draw a clear graded boundary between the performance of SSTA and FOSM control. SSTA may be classified as better for following reasons.

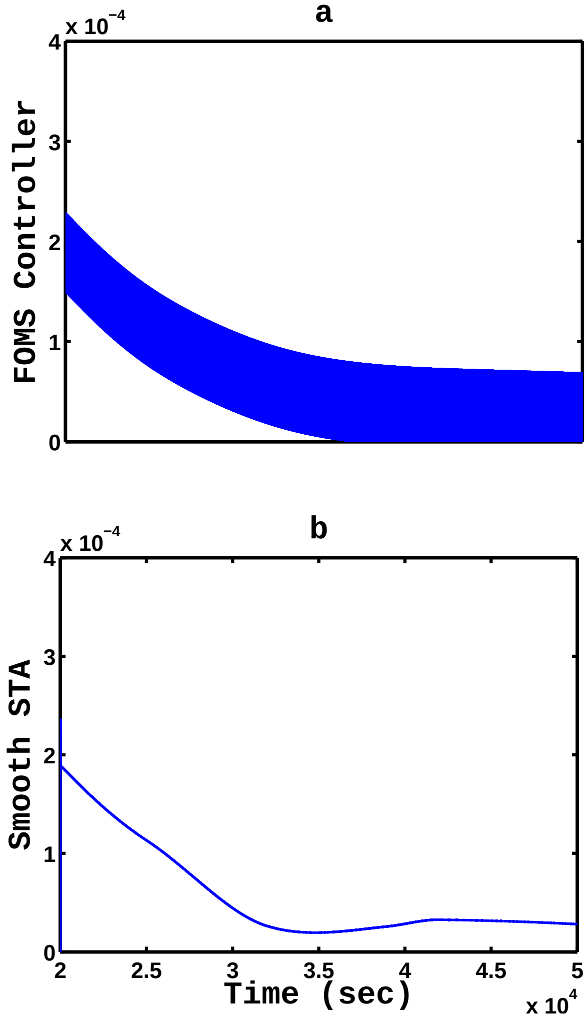


FIGURE 4.9: a) Molar Flow Rate ($moles/cm^2Sec$) (control input) produced by FOSM b) Molar Flow Rate produced by SSTA

The control effort (flow rate of the inlet gas mixture) produced by the SSTA is **Smooth** (no chattering) when compared with the high frequency oscillations in case of the FOSM control. In addition, the flow rate produced by the FOSM control disrupts the compressor limitation (constraint on the flow rate of the inlet gas mixture, see Eq. 4.44), when the water influx ζ (Figure 4.6) touches its peak value at approximately $33,000$ *secs*. Whereas, the flow rate of the inlet gas mixture, produced by the SSTA, cope with the water influx without violating the compressor limitation.

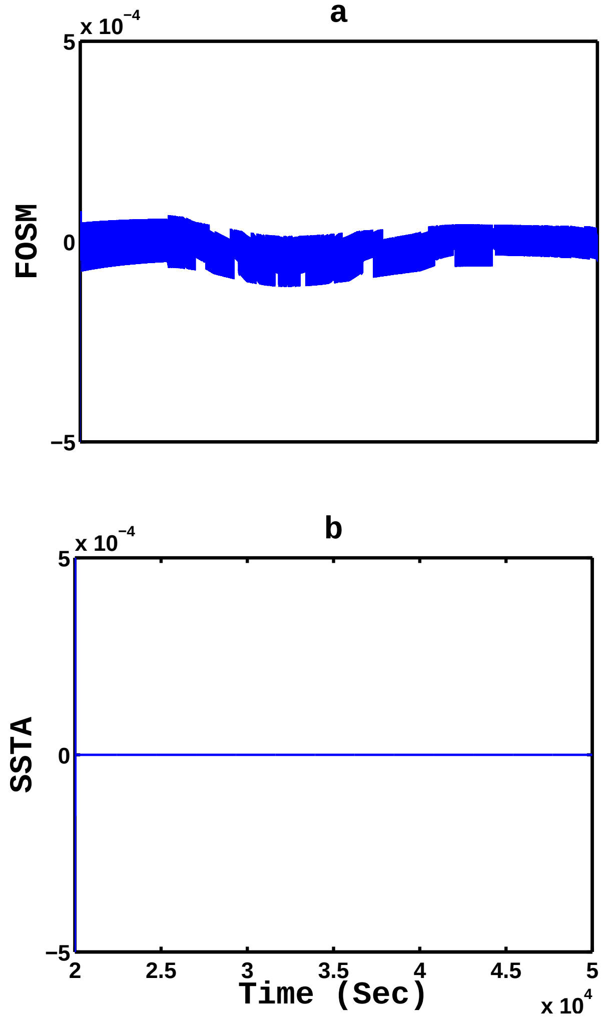


FIGURE 4.10: Sliding surface (error of output and reference) enforced by (a) FOSM (b) SSTA

The switching manifolds being reached by the FOSM and SSTA are shown in Figures 4.10(a) and 4.10(b) respectively. As mentioned earlier, the FOSM control causes saturation of the compressor (see Figure 4.9(a)) which in turn causes a noticeable effect of the water influx in the switching manifold. On the other hand, the switching manifold enforced by the SSTA is unaffected by the water influx and stays almost at 0 KJ/mol (with sliding accuracy of 10^{-9} [14]), for the whole operation.

4.5 Summary

In this chapter the analytic expressions are proposed for the gains and convergence time of the SSTA for nominal as well as the perturbed system. These objectives are achieved via the introduction of a family of quadratic strict Lyapunov functions. The analytic expressions of the convergence time reveals the importance of the smoothening parameter ρ. The SSTA converges to the origin in finite time, when it is under the effect some bounded vanishing perturbations and ensures uniform ultimate boundedness when the perturbations are non-vanishing.

This analysis can be used for many purposes, it gives the freedom to design the gains of the controller for ensuring a desired performance. The estimate of the convergence time can be subjected to some optimization framework to further improve the performance. Also, this analysis can be useful for structural improvement of the SSTA.

The SSTA with the proposed analytic expressions, is applied to the process of UCG. This is a highly nonlinear process with numerous model uncertainties. The simulation results reflect the effectiveness of the algorithm in terms of robustness and performance.

Chapter 5

IMPROVEMENTS TO INTEGRAL SLIDING MODE CONTROL

"Good behavior towards people is equivalent to wisdom, to request politely is half of knowledge, and to ascribe to sound policies is half of ones livelihood."

Hazrat Umar R.A.

A splendid feature of Sliding Mode Control (SMC) is the existence of ***sliding modes***, established after the reaching phase has been accomplished. The ***sliding modes*** are invariant with respect to uncertainties/disturbances and offer parameter invariance due to order reduction of the dynamics. On the other hand, in reaching phase the system dynamics are sensitive to disturbances/uncertainties. In [8, 68, 49] an SMC algorithm, known as the Integral Sliding Mode Control (ISMC), was proposed with the fascination of reaching phase elimination.

In ISMC the existence of sliding modes from the very beginning was established in an extended state space using an integral manifold as a surface. The extended state space eliminated the reaching phase to enhance robustness but also eradicated the order reduction property and hence any varying parameters in the nominal system[1] (see Chapter 3 for details) may seriously degrade the closed loop performance [50, 51]. In addition, the chattering phenomenon, though reduced to some extent, is still evident.

[1]The system can be treated as a nominal one because the uncertainties and disturbances has been abolished from the initial time instant.

The ISMC operates under the action of a control input, which is an algebraic sum of a continuous and a discontinuous component. In this chapter, a smooth discontinuous control component is presented, in order to generate a continuous control signal. This enabled the algorithm to be used for practical purposes, especially in a multi loop scheme in which an inner loop equipped with a PID controller takes command from an ISMC controller in the outer loop. The discontinuous component of the controller provides robustness against uncertainties and disturbances which enters the system through input channel, known as *matched*. To deal with the possible performance degradation due to parametric variations, a Linear Parameter Varying (LPV) based gain scheduling controller [69, 45, 46, 70, 71, 72] is used as the continuous part of the ISMC algorithm. In addition to providing parameter invariance, the LPV based gain scheduling controllers, by the virtue of LPV form[2], facilitated the construction of global linear controllers directly for nonlinear systems [73, 74].

The rest of this chapter is organized as follow. Section 5.1 gives a brief introduction to the problem statement. Also how this problem is handled for a SISO nonlinear/linear system, is given in this section. Section 5.2 explores the mathematical setup for the stability analysis and design of Smooth ISMC (SISMC). The design of a novel parameter dependent integral manifold is given in this section. In addition, the controller design framework, with a smooth discontinuous control component to provide robustness against uncertainties and disturbances of matched nature and an LPV based gain scheduling controller as continuous part for retaining parameter invariance, is carried out in this section. The section is concluded by rigorously stating the propositions in the form of a theorem and its proof. In Section 5.3 the proposed hybrid (LPV + Smooth ISMC) technique is

[2]In LPV form the system is not linearized about any operating point (equilibrium) rather it appears to be linear in the state variables and the coefficients vary with a signal known as the scheduling signal. The difference with time varying linear systems is that the scheduling signal can be measured and/or estimated.

applied to a laboratory test bench Ball on a Beam Balancer (BBB). The experimental results provided in this section validate the proposed algorithm in terms of robustness and performance.

5.1 Problem Formulation

Consider a SISO dynamical system, in the state space form.

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= x_4 \\
&\;\;. \\
&\;\;. \\
&\;\;. \\
\dot{x}_n &= f(t,x) + bg(t,x) + bu(t,x),
\end{aligned}
\tag{5.1}
$$

Where $x \in \Re^n$ is the state vector, $g(t,x) : \Re \times \Re^n \to \Re$ is the matched disturbance bounded by the non-negative constant η i.e., $|g(t,x)| < \eta$, $u(t,x) : \Re \times \Re^n \to \Re$ is the control input and $b \in \Re$ is the constant input channel. The term $f(t,x) : \Re \times \Re^n \to \Re$ is a linear or nonlinear function of states, which may have time or state dependent parameter/s.

The following assumptions are vital in formulation and development of the algorithm.

Assumption 4. *The disturbance $g(.)$ is norm bounded i.e.,*

$$g(t,x) \leq \eta, \quad \forall x \; and \; t \tag{5.2}$$

Assumption 5. *The possible nonlinearity and/or parametric variations in the function $f(t,x)$ can be formulated as an LPV problem to get $f(t,\rho,x)$ in such a way that $f(t,\rho,x)$ is linear in states (x) and depends affinely on scheduling parameter ρ [75, 76].*

As mentioned earlier the ISMC being famous for reaching phase elimination, generates a control input $u(t,x)$ which is the algebraic sum of a continuous part $u_0(t,x)$ and a discontinuous part $u_1(t,x)$ i.e., $u(t,x) = u_0(t,x) + u_1(t,x)$. Then based on assumption 1 and the fact that $u_1(t,x) = 0$ on the surface (as we will see later), the dynamics during *sliding* can be represented as given in Eq. 5.3.

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= x_4 \\
&. \\
&. \\
&. \\
\dot{x}_n &= f(t,\rho,x) + bu_0(t,x).
\end{aligned}
\tag{5.3}
$$

The function $f(t,\rho,x)$ is now linear in state variables $(x_1(t), x_2(t), x_3(t)......x_n(t))$ and depends affinely on the parameter ρ. Moreover ρ is upper bounded by $\overline{\rho}$ and lower bounded by $\underline{\rho}$ i.e., $\underline{\rho} \leq \rho \leq \overline{\rho}$.

The clear dependence of the dynamics in Eq. 5.3 on the parameter ρ reveals the fact that a simple state feedback linear controller or a robust H_∞ with the predefined gains/weights, for steering these sliding mode dynamics to an equilibrium, will not be able to perform well for all the values of ρ [71].

5.2 LPV Based Integral Sliding Mode Control Algorithm

The hybrid algorithm, carrying the parameter invariance property of LPV based gain scheduling controller and the robustness properties of ISMC, is developed as follows.

5.2.1 Design of Integral Manifold

The integral manifold S, for the ISMC given in [50], is of the form

$$S = S_0 + Z, \tag{5.4}$$

where $S_0 = \sum_{i=1}^{n} c_i x_i$ with $c_n = 1$, is the linear combination of states and appears as a Hurwitz and monic polynomial while Z is the integral part of the manifold which will be defined in the subsequent paragraphs.

Taking the total time derivative of Eq. 5.4 along the trajectories of Eq. 5.1 with $f(t,x)$ formulated as $f(t,\rho,x)$[3].

$$\dot{S} = \sum_{i=1}^{n-1} c_i x_{i+1} + \dot{x}_n + \dot{Z}. \tag{5.5}$$

Equating Eq. 5.5 for $\dot{Z}$ (in steady state i.e., $\dot{S} = 0$), gives the following expression for $\dot{Z}$,

$$\dot{Z} = -\sum_{i=1}^{n-1} c_i x_{i+1} - f(t,\rho,x) - bu_0(t,\rho,x). \tag{5.6}$$

The expression for $\dot{Z}$ makes the integral manifold (Eq. 5.4) dependent upon the parameter ρ and hence a gain scheduling controller is an obvious choice to be used as the continuous part of the controller i.e., $u_0(.)$ is required to be a function of ρ e.g., $u_0(t,\rho,x)$.

5.2.2 Design of Discontinuous part

The discontinuous part of the controller may be a first order or a second order SMC [68]. Loading Eq. 5.5 with $\dot{Z}$ given in Eq. 5.6,

$$\dot{S} = b(g(t,x) + u_1(t,x)). \tag{5.7}$$

Taking a smooth first order SMC as the discontinuous term:

$$u_1(t,x) = -K|S|^{\kappa}\text{sign}(b)\text{sign}(S), \tag{5.8}$$

[3] $f(t,\rho,x)$ is linear in state variables x and affinely depend upon the scheduling parameter ρ.

where $|S|^{\kappa}$ is the smoothening term and the constant κ is defined as:

$$\kappa = \begin{cases} -1/2, & \text{if } 0 < S < 1 \\ 0, & \text{if } S = 0 \\ 1/2, & \text{if } S < 0 \text{ or } S \geq 1. \end{cases} \tag{5.9}$$

The existence of sliding modes in the integral manifold can usually be computed using a Lyapunov function candidate,

$$V(t,x) = \frac{1}{2}S^2.$$

Taking the total time derivative of $V(t,x)$, with $u_1(t,x)$ given in Eq. 5.8, we have:

$$\begin{aligned} \dot{V}(t,x) &= Sb(g(t,x) - K|S|^{\kappa}\text{sign}(b)\text{sign}(S)), \\ &\leq Sb(\eta - K|S|^{\kappa}\text{sign}(b)\text{sign}(S)), \\ &\leq Sb\eta - K\,|S|^{\kappa+1}\,|b|\,. \end{aligned}$$

So if the positive non-zero constant K is chosen such that $K > \eta$ then the fact that $|S|^{\kappa+1} \geq S$ and $|b| \geq b$, reveals that $\dot{V}(t,x) \leq 0$ (semi-negative definite). This ensures the existence of *sliding modes* in the integral manifold.

5.2.3 Design of Continuous Part

The ISMC begins with the already established sliding modes and the dynamics during *sliding* represented by Eq. 5.3 while the discontinuous part $u_1(t,x)$ (see Eq. 5.8) keeps the dynamics thereafter. However, it is still needed to find an appropriate continuous part $u_0(t,\rho,x)$ to steer the dynamics (Eq. 5.3) to an equilibrium. In this regard, representing Eq. 5.3 in the following the standard state space form.

$$\dot{x}(t) = A(\rho)x(t) + Bu_0(t,\rho,x), \tag{5.10}$$

where $A(\rho) \in \Re^{n\times n}$ is a parameter dependent system matrix, $x(t) \in \Re^n$ is the state vector and $B \in \Re^n$ is the input matrix.

Remark 5.1. The continuous part ($u_0(t, \rho, x)$) of the Smooth ISMC (SISMC) is to be designed as a gain scheduling controller in order to stabilize the dynamics in Eq. 5.10 for any value of parameter ($\underline{\rho} \leq \rho \leq \overline{\rho}$).

Assumption 6. *The current value of the parameter ρ is available/measurable/observable.*

The introduction of gain scheduling state feedback controller $u_0(t, \rho, x) = M(\rho)x(t)$, where $M(\rho) \in \Re^n$ is the parameter dependent gain matrix, gives the following closed loop dynamics.

$$\begin{aligned}\dot{x}(t) &= (A(\rho) + BM(\rho))x(t),\\ &= A_{cl}(\rho)x(t).\end{aligned} \tag{5.11}$$

Where $A_{cl}(\rho)$ is the closed loop parameter dependent system matrix. The closed loop dynamics in Eq. 5.11 will be stable if there exists a common, symmetric and positive definite matrix P such that the following inequality is satisfied for all the values of the parameter ρ [77, 78].

$$\begin{aligned}PA_{cl}^T(\rho) + A_{cl}(\rho)P < 0,\\ PA^T(\rho) + A(\rho)P + PM^T(\rho)B^T + BM(\rho)P < 0.\end{aligned} \tag{5.12}$$

Note that $\rho \in co(\underline{\rho}, \overline{\rho})$ which mean that Eq. 5.11 represent a polytopic system and Eq. 5.12 represent the infinite number of LMIs [79]. So, despite solving the infinite LMIs for a common P, it is worthy to solve them only at the vertices of the convex hull.

Theorem 5.2. *With $u_1(t, x)$ given in Eq. 5.8, if the inequality in Eq. 5.12 is satisfied with a common symmetric and positive definite matrix P at the vertices of the convex hull $co(\underline{\rho}, \overline{\rho})$ then the dynamics in Eq. 5.1 will exhibit convergent sliding modes. Moreover, if the following LMI,*

$$L \otimes P + N \otimes A_{cl}(\rho)P + N^T \otimes (A_{cl}(\rho)P)^T < 0, \tag{5.13}$$

is satisfied for a symmetric and positive definite matrix P then Eq. 5.3 will be D-stable and will have poles in the D-region defined by the matrices L and N.

Proof. The first part of the theorem has already been proved but is mentioned here for the sake of completeness. Taking a positive definite and radially unbounded Lyapunov function candidate,

$$V(t,x) = \frac{1}{2}S^2.$$

The total time derivative of $V(.)$ along the trajectories of Eq. 5.1 and using equations 5.4, 5.6, 5.7 and 5.8, we have,

$$\begin{aligned}\dot{V}(t,x) &= Sb\left(g(t) + u_1(t)\right),\\ &\leq Sb\left(\eta + u_1(t)\right),\\ &\leq Sb\left(\eta - K|S|^{\kappa}\text{sign}(b)\text{sign}(S)\right),\\ &\leq Sb\eta - K|S|^{\kappa+1}\left|b\right|.\end{aligned}$$

As $S \leq |S|^{\kappa+1}$, $b \leq |b|$ and $\eta < K$, so

$$\dot{V}(t,x) \leq -\left|Sb\eta - k|S|^{\kappa+1}\left|b\right|\right| \leq 0, \tag{5.14}$$

which confirms the existence of sliding modes.

For the proof of second part, solving the LMI in Eq. 5.13, at the vertices of the convex hull, for a common symmetric and positive definite matrix P. In addition, let v be the left eigenvector of $A_{cl}(\rho)$ corresponding to eigen value λ, then:

$$v^H A_{cl}(\rho) = \lambda v^H,$$

and by congruence transformation the inequality (Eq. 5.13) is presented as:

$$\begin{aligned}(I \otimes v^H)[L \otimes P + N \otimes A_{cl}(\rho)P...\\ +N^T \otimes (A_{cl}(\rho)P)^T](I \otimes v) < 0,\\ L \otimes v^H P v + N \otimes v^H A_{cl}(\rho) P v...\\ +N^T \otimes v^H P A_{cl}^T(\rho) v < 0,\\ L + \lambda N + \lambda^* N^T < 0,\end{aligned} \tag{5.15}$$

where λ^* represent the complex conjugate of eigen value λ. The inequality in Eq. 5.15, defines the sufficient condition for the eigen values of $A_{cl}(\rho)$ to be in the LMI region characterized by L and N [79]. □

The algebraic expression for the scheduled gain $M(\rho)$ can be found directly by evaluating Eq. 5.13 at the vertices.

$$L \otimes P + N \otimes (A(\underline{\rho}) + BM(\underline{\rho}))P...$$

$$+N^T \otimes ((A(\underline{\rho}) + BM(\underline{\rho}))P)^T < 0, \tag{5.16}$$

where $M(\underline{\rho})$ is the scheduled controller gain for the system characterized by vertex $\underline{\rho}$. The inequality in Eq. 5.16 is nonlinear in the variables $M(\underline{\rho})$ and P. In order to make it linear a new variable $\psi(\underline{\rho}) = M(\underline{\rho})P$ is introduced.

$$L \otimes P + N \otimes (A(\underline{\rho})P + B\psi(\underline{\rho}))...$$

$$+N^T \otimes ((A(\underline{\rho})P + B\psi(\underline{\rho}))^T < 0. \tag{5.17}$$

Solution of Eq. 5.17 for P and $\psi(\underline{\rho})$ gives $M(\underline{\rho}) = \psi(\underline{\rho})P^{-1}$. Similarly the gain $M(\overline{\rho})$ can be found from Eq. 5.17 using the same P.

The final scheduled controller gain $M(\rho)$ can be obtained from the algebraically weighted convex combination of $M(\underline{\rho})$ and $M(\overline{\rho})$.

$$M(\rho) = r_1 M(\overline{\rho}) + r_2 M(\underline{\rho}),$$

where r_1 and r_2 are constants weighting functions such that they ensure the convex combination of $M(\underline{\rho})$ and $M(\overline{\rho})$ i.e.,

$$r_1 + r_2 = 1.$$

These constants has the following mathematical representations.

$$r_1 = \frac{\overline{\rho} - \rho}{\overline{\rho} - \underline{\rho}},$$

$$r_2 = \frac{\rho - \underline{\rho}}{\overline{\rho} - \underline{\rho}}.$$

In the next section the proposed algorithm is applied to the laboratory test bench, Ball on a Beam Balancer (BBB). BBB system has inherent nonlinearity in control input. The conventional linear approximations of this system fails due to practically invalid assumption of *input being small enough*. The proposed algorithm does not actually linearize the system during sliding rather, by the virtue of LPV form, makes it *look linear*, in state variables, for controller design purposes.

5.3 Ball On A Beam Balancer

Figure 5.1: Experimental Test Bench

The *Ball on a Beam balancer* is considered as an important test bench in the field of control engineering because of the wider spectrum in the form of nonlinearity and inherent open loop instability. The major task of this test bench is to control the position of a stainless steel ball on a metallic beam.

5.3.1 Physical Description

The Ball on a Beam Balancer is comprised of a metallic beam, a gear assembly and a DC servo motor. The left most end of the metallic beam is fixed while the right most end can be stimulated for up and down motion by means of the DC servo motor and the gear assembly. The position of the ball on the metallic beam is measured through voltage variations, created by the ball movement, across the metallic beam. The angular position of the spindle of the DC servo motor is measured by an absolute potentiometer. The control signal generated by an interfaced computer is given to DC servo motor via a power amplifier.

5.3.2 Mathematical Description

The simplified mathematical model of the Ball on a Beam Balancer is reported as two coupled systems (see for details [80] and www.quanser.com) i.e., the dynamics of a DC servo motor,

$$\ddot{\theta}_l(t) = -\frac{B_{eq}}{J_{eq}}\dot{\theta}_l(t) + \frac{\gamma}{J_{eq}}V(t), \tag{5.18}$$

and the dynamics of ball moving over the beam.

$$\ddot{x}(t) = \frac{5rg}{7L}\sin(\theta_l). \tag{5.19}$$

It may be observed that the controller has to put effects on the DC servo motor in order to control the ball position (see only θ_l appearing in Eq. 5.19). The variables and parameters in Eq. 5.19 and Eq. 5.18 are listed in Table 5.1.

5.3.3 Problem Description

The control objective is to operate the DC Servo motor in such a way that the ball is robustly kept at the beam center. This is accomplished by two feedback loops and hence two different controllers as shown in Figure 5.2.

The outer loop, which serves as a *guide* for the inner loop, is equipped with the proposed hybrid Smooth Integral Sliding Mode Control (SISMC) algorithm, which

Entity	Notation	Value	Unit
Beam Length	L	41	cm
Lever Arm Offset	r	2.5	cm
Servo Gear Angle	θ_l	± 60	deg
Moment of Inertia	J_{eq}	2.084×10^{-3}	$kg.m^2$
Back EMF Constant	C_1	7.68×10^{-3}	$V.Sec/rad$
Damping co-efficient	B_{eq}	8.40×10^{-2}	-
Torque per unit voltage	γ	0.1285	$N.m/volt$
Control Voltage	V	-	$Volts$
Gravitation Constant	g	9.81	m/sec^2
Ball Position	x	-0.2 to 0.2	m

TABLE 5.1: Physical Specifications

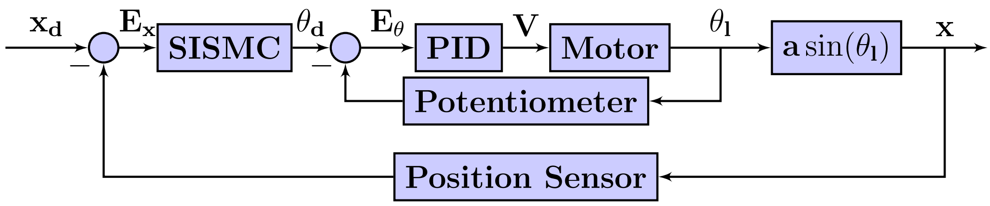


FIGURE 5.2: Control Configuration for Ball on a Beam Balancer

takes into account the error/difference between the desired and current position of the ball on the metallic beam and generates an output which actually becomes a desired angle/reference for the inner loop (DC servo motor). A PID controller in the inner loop takes this reference angle from the SISMC and provides a controlled voltage to the DC servo motor such that $\theta_l = \theta_d$ which implies $x = x_d$.

Remark 5.3. The smoothness of SISMC implies that the continuous PID will not be destabilized/detracted due to any discontinuity/chattering in the reference angle, which would be the case if the ISMC was not *smooth*.

The main problem in designing a linear controller for the system in Eq. 5.19 is that $\sin(\theta_l)$ cannot be approximated equivalent to θ_l because the variations in θ_l are not small (see Table 5.1)[4].

In this work the nonlinearity in control input is handled such that the system appears linear in state variables and control input. Let us define a scheduling parameter $\rho = \sin(\theta_l)/(\theta_l + \epsilon)$, where $\epsilon \to 0$. Figure 5.3 shows a plot of ρ for range of θ_l given in Table 5.1.

With the above description of ρ and for a constant $a = 5rg/7L$, the dynamics of the ball (see Eq. 5.19) can be represented as:

$$\ddot{x}(t) = a\rho\theta_l,$$

[4]Linear approximation failed due to certain physical characteristics.

with the corresponding state space,

$$\dot{x}_1 = x_2,$$

$$\dot{x}_2 = a\rho\theta_l.$$

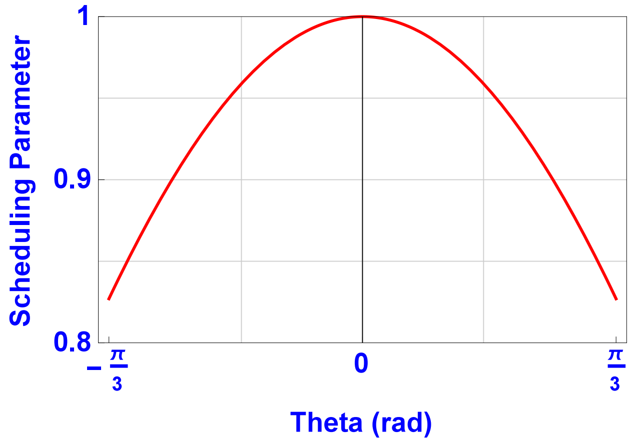


FIGURE 5.3: Scheduling Parameter

5.3.4 Experimental Results

The experimental results are presented with the formulas, values and softwares given in Table 5.2.

The ball position being controlled by the SISMC is shown in Figure 5.4. It may be observed that the ball position is maintained at the center of the metallic beam by the combination of SISMC and PID, in less than 10 *seconds*. The robustness of the SISMC may also be observed from this figure. The ball has been disturbed (perturbed from the center) at around 11^{th} second and 30^{th} second of the experiment. In both the cases the ball effectively comes back to the center of the beam. The zoomed view in this figure (Figure 5.4) shows the accuracy of the SISMC algorithm. It may also be noticed that the negative values on position axis are just for differentiating the left and right side of the metallic beam with respect to its center.

Notation	Value/Range
S	$S_0 + Z$
S_0	$\dot{e} + Ce$
Z	$-\int(a\rho u_0 + C\dot{e})dt$
e	$x_d - x$
$\overline{\rho}$	1
$\underline{\rho}$	0.78
$M(\overline{\rho})$	$\begin{bmatrix}20 & 10\end{bmatrix}$
$M(\underline{\rho})$	$\begin{bmatrix}-20 & -10\end{bmatrix}$
u_0	$M(\rho)\begin{bmatrix}e & \dot{e}\end{bmatrix}^T$
u_1	$-K\lvert S\rvert^{\kappa} Sign(S)Sign(\rho)$
θ_d	$u_0 + u_1$
K	0.4
C	0.5
MATLAB/Simulink	7.10
Visual Studio	2008
Quanser Real Time Control	QuaRC
Mathematica	10
Step Time	0.001 Sec

Table 5.2: Controller Parameters and Software Specifications

In Figure 5.5 the control effort (θ_d) (which is reference for the inner loop i.e., DC servo motor) produced by the SISMC and the corresponding motor angle (θ_l) (angular position of the spindle), are shown. The controller is effectively responding to any change in ball position, keeping the ball at the beam center and importantly without causing any saturation to the DC motor. In Figure 5.6 the controlled voltage ($V(t)$) produced by the PID controller, such that θ_l tracks θ_d, is shown. It may be observed that the smoothness of the SISMC leaves the performance of the PID un-affected[5].

[5]A discontinuity in the sliding mode controller will cause chattering in θ_d. Due to this discontinuity the performance of the continuous PID controller may degrade to a huge extent.

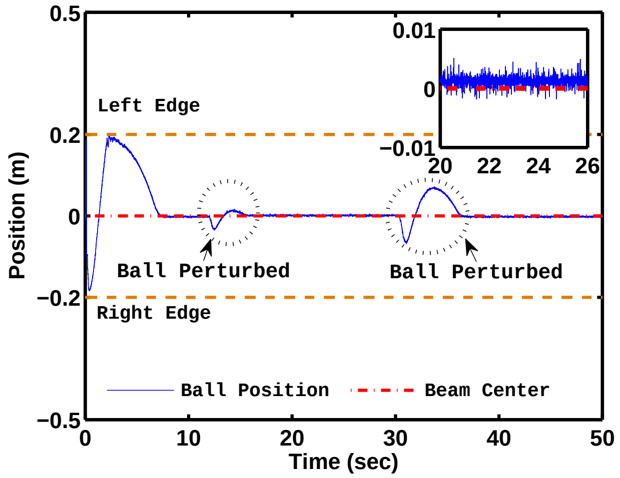


FIGURE 5.4: Ball Position

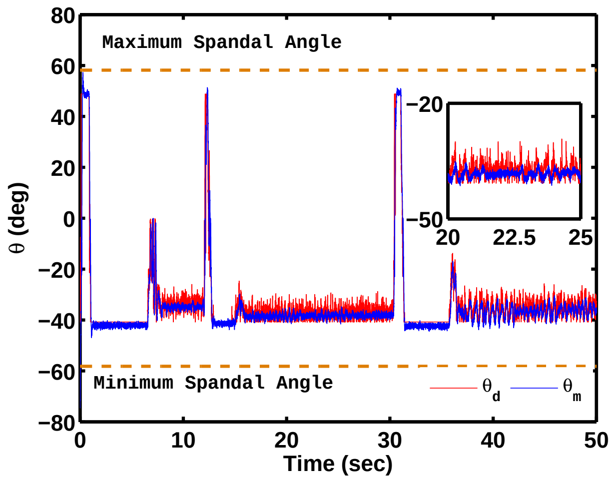


FIGURE 5.5: Desired and Measured angles for DC Servo Motor

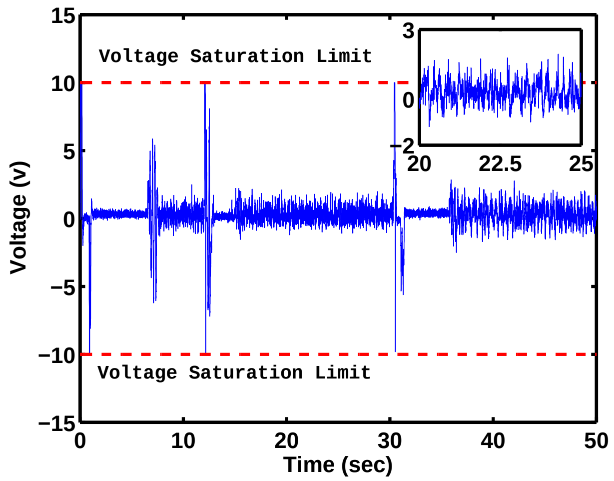


FIGURE 5.6: Voltage Applied to the DC Motor

Figure 5.7 shows the integral manifold (S) achieved by the SISMC. The initial short reaching phase is solely due to the hardware calibration and achieving pin point accurate initial condition for the integral term. However, the proposed SISMC is effective enough to compensate this short reaching phase. The zoomed portion in this figure show that the algorithm's sliding accuracy is almost 10^{-4}, which is very near to an ideal sliding definition.

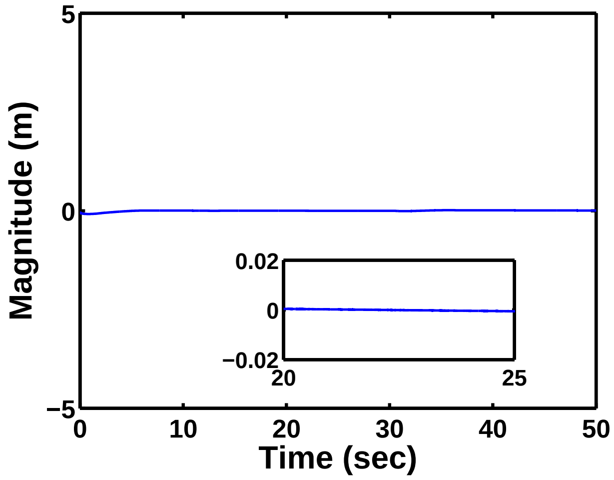


FIGURE 5.7: Sliding Surface

5.4 Summary

The reaching phase elimination property of the ISMC, good for robustness and performance, subtracts a useful advantage of order reduction from this algorithm. As a result the performance of the sliding mode dynamics may be sensitive to parametric variations. This problem is addressed via modification to the integral manifold and using an LPV based gain scheduling controller as the continuous part of the controller. The continuous controller, addresses the problem of performance degradation due to parametric variations, operates in combination with a discontinuous controller. The discontinuous part of the controller, will provide robustness against bounded matched disturbances and uncertainties, and will keep the dynamics on the integral manifold. In addition, the discontinuous part of the proposed ISMC algorithm is made smooth in order to produce a chattering free controller action. The experimental results obtained from the laboratory test bench, ball on a beam balancer, show the effectiveness of the proposed strategy.

Chapter 6

CONCLUSIONS AND FUTURE WORK

"Twenty years from now you will be more disappointed by the things that you didn't do than by the ones you did do. So throw off the bowlines. Sail away from the safe harbor. Catch the trade winds in your sails. Explore. Dream. Discover."

H. Jackson Browne.

This chapter summarizes the idea contained in the thesis. A discussion on possible future research directions is also initiated for the community working on sliding mode control theory and applications.

6.1 Featured Highlights

Sliding Mode Control (SMC), powered by the application of a discontinuous controller and the existence of sliding motion, is composed of two phases, **reaching phase** and **sliding phase**. The SMC provided guaranteed robustness, against uncertainties and disturbances, for a huge variety of control problems. In addition, it offers order reduction and hence parameter invariance, in the reaching phase. However, the SMC do possess some inherent limitations. The most pronounced one is the existence of very high frequency oscillations, during sliding phase, termed as **chattering**. The other problems include the sensitivity to the uncertainties and disturbances in the reaching phase and the relative degree requirement.

The problem of chattering is well addressed by the use of Higher Order Sliding Mode (HOSM) control such as Super Twisting Algorithm (STA) and Real Twisting Algorithm (RTA). The STA and RTA are reported very sensitive to the un-modeled fast dynamics of the underlying system due to which chattering may appear sooner or later in the closed loop. The introduction of Smooth Sliding Mode (SSM) control, such as Smooth STA (SSTA) and Smooth RTA (SRTA), coped with this problem at the cost of robustness. The SSTA however, is robustified with the use of a second order sliding modes based disturbance observer. In addition, the performance of the algorithm is not parameterized in terms of its parameters.

The sensitivity to disturbances in the reaching phase is effectively coped with using the Integral Sliding Mode (ISM) control. The ISM control, by the virtue of a new type of sliding manifold known as the integral manifold, eliminated the reaching phase out of the SMC at the cost of no order reduction in the sliding phase. The fact of no order reduction, in general, eliminated the property of parameter invariance. This may cause performance degradation in the closed loop. In addition, the control input of ISM control algorithm is an algebraic sum of a discontinuous and a continuous controller. The discontinuous term in usually the conventional sliding mode controller which will cause chattering effects, though reduced, but still limit its use in sensitive applications.

The performance based analysis and design of the Smooth Sliding Mode Control (SSMC) algorithms has been presented in this thesis. The continuous nature of the SSMC algorithms, make them a safe controller design paradigm, especially when the system to be controlled is very sensitive and the performance and robustness is under critical consideration.

The smooth (continuous) Higher Order Sliding Modes (HOSM) based on Super Twisting Algorithm (STA), is parameterized, in terms of performance and robustness. The parametrization of the algorithm carries with itself various advantages such as: Structural improvements can be made, performance can be set in terms of controller parameters and also explores the effects of different controller parameters on the robustness and performance of the closed loop system. In addition, the analytical expressions for the gains of SSTA gives a best starting point for further design iterations. The proposed analytical expressions for the controller gains, guarantee robustness without employing a disturbance observer.

The proposed design of the SSTA is tested against the process control of the Underground Coal Gasification (UCG) process, which is a highly complex nonlinear system.

The thesis also contains a modified Integral Sliding Mode Control (ISMC), equipped with a Linear Parameter Varying (LPV) based gain scheduling controller as its continuous part. The proposed ISMC, by the virtue of continuous gain scheduling controller cope with any possible performance degradation in the presence of parametric variations. In addition, the smoothening effects are incorporated in the design of the discontinuous part, that is why we call it Smooth ISMC (SISMC). The incorporated smoothening effect in the overall algorithm allow using it in any sensitive application because of the continuous controller action (no chattering). One such situation is a multi loop scheme, especially when a continuous inner loop controller gets signal from an outer loop controller. In contrast, the incorporated LPV form give liberty of designing global linear controllers for the the systems which may not be possible to be linearized by conventional means.

The proposed SISMC is experimentally tested on a bench Ball on a Beam Balancer (BBB). The BBB is composed of two coupled nonlinear systems, such as the ball rolling over a metallic beam and a servo DC motor which gyrates the metallic beam. The input from the DC servo motor appears nonlinear to the beam. More importantly this input can not be linearized by the conventional means. The proposed SISMC not only cope with the problem of linearization but also produces a response in stabilizing the ball over the beam.

6.2 Future Research Directions

The research work in this thesis focuses on the ***"Smoothness"*** of Sliding Mode Controllers and their ***"Performance"*** in closed loop. However, the proposed work can be extended from theoretical as well as from application perspectives.

In theoretical point of view the possible extension are listed below.

- Structural improvements can be made to the existing Smooth Sliding Mode Control Algorithms.
- The analysis can be extended to the theory to Multi-Input-Multi-Output (MIMO) sliding modes.
- MIMO integral sliding modes with more than one parameters may be developed.
- The SISMC algorithm may be researched for the case when the varying parameters are not available/measurable e.g., incorporating an LPV observer in the control loop.
- A general theory of rubustified LPV compensation may be developed based on integral sliding mode control.
- Modifications can be proposed in these algorithms for a variety of system classes such as under actuated system and non-holonomic systems.
- The Smooth Super Twisting Algorithm (SSTA) with the proposed parametrization can be subjected to some optimization routine for performance improvement and optimality.

Some possible application perspectives may be as follows:

- Robust LPV estimation of thermally de-rated torque of a Hybrid Electric Vehicle (HEV) drive may be carried out with the hybrid SISMC.
- Control of a two wheeled robot, which offers the type of motion similar to the ball on a beam balancer presented in this thesis, may be another engineering test bench.
- The take-off and landing control problem of Unmanned Ariel Vehicle (UAV), with inherent parametric variations, can be an obvious and challenging benchmark for the proposed SISMC.

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